GRADUATE RECORD EXAMINATIONS®

Math Review

Chapter 2: Algebra

The GRE® Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis. This is the accessible electronic format (Word) edition of the Algebra Chapter of the Math Review. Downloadable versions of large print (PDF) and accessible electronic format (Word) of each of the 4 chapters of the of the Math Review, as well as a Large Print Figure supplement for each chapter are available from the GRE® website. Other downloadable practice and test familiarization materials in large print and accessible electronic formats are also available. Tactile figure supplements for the four chapters of the Math Review, along with additional accessible practice and test familiarization materials in other formats, are available from E T S Disability Services Monday to Friday 8:30 a m to 5 p m New York time, at 1-6 0 9-7 7 1-7 7 8 0, or 1-8 6 6-3 8 7-8 6 0 2 (toll free for test takers in the United States, U S Territories and Canada), or via email at stassd@ets.org.

The mathematical content covered in this edition of the Math Review is the same as the content covered in the standard edition of the Math Review. However, there are differences in the presentation of some of the material. These differences are the result of adaptations made for presentation of the material in accessible formats. There are also slight differences between the various accessible formats, also as a result of specific adaptations made for each format.

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Mathematical Equations and Expressions

The Math Review includes mathematical equations and expressions. In accessible electronic format (Word) editions some of the mathematical equations and expressions are presented as graphics. In cases where a mathematical equation or expression is presented as a graphic, a verbal presentation is also given and the verbal presentation comes directly after the graphic presentation. The verbal presentation is in green font to assist readers in telling the two presentation modes apart. Readers using audio alone can safely ignore the graphical presentations, and readers using visual presentations may ignore the verbal presentations.

Table of Contents

Overview of the Math Review

The Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis.

Each of the 4 chapters in the Math Review will familiarize you with the mathematical skills and concepts that are important to understand in order to solve problems and reason quantitatively on the Quantitative Reasoning measure of the GRE® revised General Test.

The material in the Math Review includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each chapter. Note, however that this review is not intended to be all inclusive. There may be some concepts on the test that are not explicitly presented in this review. If any topics in this review seem especially unfamiliar or are covered too briefly, we encourage you to consult appropriate mathematics texts for a more detailed treatment.

Overview of this Chapter

Basic algebra can be viewed as an extension of arithmetic. The main concept that distinguishes algebra from arithmetic is that of a **variable**, which is a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any letter can be used. Variables enable you to present a word problem in terms of unknown quantities by using algebraic expressions, equations, inequalities, and functions. This chapter reviews these algebraic tools and then progresses to several examples of applying them to solve real life word problems. The chapter ends with coordinate geometry and graphs of functions as other important algebraic tools for solving problems.

2.1 Operations with Algebraic Expressions

An **algebraic expression** has one or more variables and can be written as a single **term** or as a sum of terms. Here are four examples of algebraic expressions.

Example A: $2x$

Example B: $y - \frac{1}{4}$ *y* minus, one fourth

Example C: $w^3z + 5z^2 - z^2 + 6$ *w* cubed *z*, +, 5, *z* squared, minus *z* squared, +, 6

Example D: $\frac{8}{n + p}$ the expression with numerator 8 and denominator *n* + *p*

In the examples above, $2x$ is a single term,

$y - \frac{1}{4}$ *y* minus, one fourth has two terms,

$w^3z + 5z^2 - z^2 + 6$ *w* cubed *z*, +, 5, *z* squared, minus *z* squared, +, 6 has four terms, and

$\frac{8}{n + p}$ the expression with numerator 8 and denominator *n* + *p* has one term.

In the expression $w^3z + 5z^2 - z^2 + 6,$ *w* cubed *z*, +, 5, *z* squared, minus *z* squared, +, 6,

the terms $5z^2$ and $-z^2$ 5, *z* squared, and negative, *z* squared

are called **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called a **constant** term.

A number that is multiplied by variables is called the **coefficient** of a term. For example, in the expression

$2x^2 + 7x - 5$, 2, *x* squared, +, 7*x*, minus 5,

2 is the coefficient of the term $2x^2$, 2, *x* squared,

7 is the coefficient of the term $7x$, and

-5 negative 5 is a constant term.

The same rules that govern operations with numbers apply to operations with algebraic expressions. One additional rule, which helps in simplifying algebraic expressions, is that like terms can be combined by simply adding their coefficients, as the following three examples show.

Example A: $2x + 5x = 7x$

Example B: $w^3z + 5z^2 - z^2 + 6 = w^3z + 4z^2 + 6$

w cubed *z*, +, 5, *z* squared, minus *z* squared, +, 6 = *w* cubed *z*, +, 4, *z* squared, +, 6

Example C: $3xy + 2x - xy - 3x = 2xy - x$

3 *x y*, +, 2*x*, minus *x y*, minus 3*x* = 2 *x y*, minus *x*

A number or variable that is a factor of each term in an algebraic expression can be factored out, as the following three examples show.

Example A: $4x + 12 = 4(x + 3)$

4*x* + 12 = 4 times, open parenthesis, *x* + 3, close parenthesis

Example B: $15y^2 - 9y = 3y(5y - 3)$

15, *y* squared, minus 9*y*, =, 3*y* times, open parenthesis, 5*y* minus 3, close parenthesis

Example C: For values of *x* where it is defined, the algebraic expression $\frac{7x^2 + 14x}{2x + 4}$

with numerator 7, *x* squared, +, 14*x* and denominator 2*x*, +, 4
can be simplified as follows.

First factor the numerator and the denominator to get $\frac{7x(x + 2)}{2(x + 2)}$.

the algebraic expression with numerator 7*x* times, open parenthesis, *x* + 2, close parenthesis, and denominator 2 times, open parenthesis, *x* + 2, close parenthesis.

Now, since *x* + 2 occurs in both the numerator and the denominator, it can be canceled out when $x + 2 \neq 0$, *x* + 2 is not equal to 0, that is, when $x \neq -2$ *x* is not equal to negative 2 (since division by 0 is not defined). Therefore, for all $x \neq -2$, *x* not equal to negative 2, the expression is equivalent to $\frac{7x}{2}$. 7*x* over 2.

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second expression, and the results are added, as the following example shows.

To multiply

$(x + 2)(3x - 7)$

open parenthesis, *x* + 2, close parenthesis, times, open parenthesis 3*x* minus 7, close parenthesis

first multiply each term of the expression *x* + 2 by each term of the expression $3x - 7$ 3*x* minus 7 to get the expression

$x(3x) + x(-7) + 2(3x) + 2(-7)$.

x times 3*x*, +, *x* times negative 7, +, 2 times 3*x*, +, 2 times negative 7.

Then multiply each term to get

$3x^2 - 7x + 6x - 14$.

3, *x* squared, minus 7*x*, +, 6*x*, minus 14.

Finally, combine like terms to get

$3x^2 - x - 14$.

3, *x* squared, minus *x*, minus 14.

So you can conclude that $(x + 2)(3x - 7) = 3x^2 - x - 14$.

open parenthesis, *x* + 2, close parenthesis, times, open parenthesis 3*x* minus 7, close parenthesis = 3, *x* squared, minus *x*, minus 14.

A statement of equality between two algebraic expressions that is true for all possible values of the variables involved is called an **identity**. All of the statements above are identities. Here are three standard identities that are useful.

Identity 1: $(a + b)^2 = a^2 + 2ab + b^2$

open parenthesis, *a* + *b*, close parenthesis, squared, =, *a* squared, +, 2 *a b*, +, *b* squared

Identity 2: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

open parenthesis, *a* minus *b*, close parenthesis, cubed, =, *a* cubed, minus 3, *a* squared *b*, +, 3*a b* squared, minus *b* cubed

Identity 3: $a^2 - b^2 = (a + b)(a - b)$

a squared minus *b* squared = open parenthesis, *a* + *b*, close parenthesis, times, open parenthesis, *a* minus *b*, close parenthesis

All of the identities above can be used to modify and simplify algebraic expressions. For example, identity 3,

$a^2 - b^2 = (a + b)(a - b)$,

a squared minus *b* squared = open parenthesis, *a* + *b*, close parenthesis, times, open parenthesis, *a* minus *b*, close parenthesis

can be used to simplify the algebraic expression $\dfrac{x^2 - 9}{4x - 12}$

with numerator *x* squared minus 9 and denominator 4*x*, minus 12

as follows.

$$\frac{x^2 - 9}{4x - 12} = \frac{(x + 3)(x - 3)}{4(x - 3)}.$$

the algebraic expression with numerator *x* squared minus 9 and denominator 4*x* minus 12 = the algebraic expression with numerator, open parenthesis, *x* + 3, close parenthesis, times, open parenthesis, *x* minus 3, close parenthesis, and denominator 4 times, open parenthesis, *x* minus 3, close parenthesis.

Now, since $x - \mathbf{3}$ *x* minus 3 occurs in both the numerator and the denominator, it can be canceled out when $x - 3 \neq 0$; *x* minus 3 is not equal to 0, that is, when $x \neq 3$ *x* is not equal to 3 (since division by 0 is not defined). Therefore, for all $x \neq 3$, *x* not equal to 3, the expression is equivalent to $\frac{x+3}{4}$. the expression with numerator x + 3 and denominator 4.

A statement of equality between two algebraic expressions that is true for only certain values of the variables involved is called an **equation**. The values are called the **solutions** of the equation.

The following are three basic types of equations.

Type 1: A **linear equation in one variable**: for example, $3x + 5 = -2$

3*x* + 5 = negative 2

Type 2: A **linear equation in two variables**: for example, $x - 3y = 10$

x minus 3*y* = 10

Type 3: **A quadratic equation in one variable**: for example $20y^2 + 6y - 17 = 0$

20, *y* squared, +, 6*y* minus 17 = 0

2.2 Rules of Exponents

In the algebraic expression x^a, *x* superscript *a*, where *x* is raised to the power *a*, *x* is called a **base** and *a* is called an **exponent**. Here are seven basic rules of exponents, where the bases x and y are nonzero real numbers and the exponents a and b are integers.

Rule 1: $x^{-a} = \frac{1}{x^a}$

x to the power negative *a* = 1, over, *x* to the power *a*

Example A: $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

4 to the power negative 3 = 1, over, 4 to the power 3, which is equal to 1 over 64

Example B: $x^{-10} = \frac{1}{x^{10}}$

x to the power negative 10 = 1 over, *x* to the power 10

Example C: $\frac{1}{2^{-a}} = 2^a$

1, over, 2 to the power negative *a* = 2 to the power *a*

Rule 2: $\left(x^a\right)\left(x^b\right) = x^{a+b}$

Open parenthesis, *x* to the power *a*, close parenthesis, times, open parenthesis, *x* to the power *b*, close parenthesis, =, *x* to the power *a* + *b*

Example A: $\left(3^2\right)\left(3^4\right) = 3^{2+4} = 3^6 = 729$

Open parenthesis, 3 squared, close parenthesis, times, open parenthesis, 3 to the power 4, close parenthesis, =, 3 to the power 2 + 4, which is equal to 3 to the power 6, or 729

Example B: $\left(y^3\right)\left(y^{-1}\right) = y^2$

Open parenthesis, *y* cubed, close parenthesis, times, open parenthesis, *y* to the power negative 1, close parenthesis =, *y* squared

Rule 3: $\dfrac{x^a}{x^b} = x^{a-b} = \dfrac{1}{x^{b-a}}$

x to the power *a*, over, *x* to the power *b*, =, *x* to the power *a* minus *b*, which is equal to 1 over, *x* to the power, *b* minus *a*

Example A: $\dfrac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$

5 to the power 7, over, 5 to the power 4, =, 5 to the power 7 minus 4, which is equal to 5 to the power 3, or 125

Example B: $\dfrac{t^3}{t^8} = t^{-5} = \dfrac{1}{t^5}$

t to the power 3, over, *t* to the power 8, =, *t* to the power negative 5, which is equal to 1, over, *t* to the power 5

Rule 4: $x^0 = 1$ *x* to the power 0 = 1

Example A: $7^0 = 1$ 7 to the power 0 = 1

Example B: $(-3)^0 = 1$

open parenthesis, negative 3, close parenthesis, to the power 0, =, 1

Note that 0^0 0 to the power 0 is not defined.

Rule 5: $\left(x^a\right)\left(y^a\right) = (xy)^a$

Open parenthesis, *x* to the power *a*, close parenthesis, times, open parenthesis, *y* to the power *a*, close parenthesis, =, open parenthesis, *x y*, close parenthesis, to the power *a*

Example A: $\left(2^3\right)\left(3^3\right) = 6^3 = 216$

Open parenthesis, 2 to the power 3, close parenthesis, times, open parenthesis, 3 to the power 3, close parenthesis, =, 6 to the power 3, or 216

Example B: $(10z)^3 = 10^3z^3 = 1{,}000z^3$

Open parenthesis, 10*z*, close parenthesis, cubed, =, 10 cubed times *z* cubed, which is equal to 1,000, *z* cubed

Rule 6: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Open parenthesis, *x* over *y*, close parenthesis, to the power *a*, =, *x* to the power *a*, over *y* to the power *a*

Example A: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

Open parenthesis, 3 fourths, close parenthesis, squared, =, 3 squared over 4 squared, which is equal to 9 over 16

Example B: $\left(\frac{r}{4t}\right)^3 = \frac{r^3}{64t^3}$

Open parenthesis, *r* over 4*t*, close parenthesis, cubed, = *r* cubed, over, 64, *t* cubed

Rule 7: $\left(x^a\right)^b = x^{ab}$

Open parenthesis, *x* to the power *a*, close parenthesis, to the power *b*, =, *x* to the power *a b*

Example A: $\left(2^5\right)^2 = 2^{10} = 1{,}024$

Open parenthesis, 2 to the power 5, close parenthesis, squared, =, 2 to the power 10, which is equal to 1,024

Example B: $\left(3y^6\right)^2 = \left(3^2\right)\left(y^6\right)^2 = 9y^{12}$

Open parenthesis, 3, *y* to the power 6, close parenthesis, squared, =, open parenthesis, 3 squared, close parenthesis, times, open parenthesis, *y* to the power 6, close parenthesis, squared, which is equal to 9, *y* to the power 12

The rules above are identities that are used to simplify expressions. Sometimes algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. In order to avoid mistakes commonly made when dealing with exponents keep the following six cases in mind.

Case 1: $x^a y^b \neq (xy)^{a+b}$

x to the power *a* times *y* to the power *b* is not equal to, open parenthesis, *x y*, close parenthesis, to the power *a* + *b*

Note that in the expression $x^a y^b$ *x* to the power *a* times *y* to the power *b* the bases are not the same, so Rule 2, $\left(x^a\right)\left(x^b\right) = x^{a+b}$,

open parenthesis, *x* to the power *a*, close parenthesis, times, open parenthesis, *x* to the power *b*, close parenthesis, =, *x* to the power *a* + *b*,
does not apply.

Case 2: $\left(x^a\right)^b \neq x^a x^b$

Open parenthesis, *x* to the power *a*, close parenthesis, to the power *b* is not equal to, *x* to the power *a* times *x* to the power *b*

Instead, $\left(x^a\right)^b = x^{ab}$

Open parenthesis, *x* to the power *a*, close parenthesis, to the power *b*, =, *x* to the power *a b*

and $x^a x^b = x^{a+b}$;

x to the power *a* times *x* to the power *b*, =, *x* to the power *a* + *b*;

for example, $\left(4^2\right)^3 = 4^6$ and $4^2 4^3 = 4^5$.

open parenthesis, 4 squared, close parenthesis, cubed, =, 4 to the power 6, and 4 squared times 4 cubed, =, 4 to the power 5.

Case 3: $(x + y)^a \neq x^a + y^a$

open parenthesis, *x* + *y*, close parenthesis, to the power *a*, is not equal to *x* to the power *a*, +, *y* to the power *a*

Recall that $(x + y)^2 = x^2 + 2xy + y^2$;

open parenthesis, *x* + *y*, close parenthesis, squared, =, *x* squared, +, 2*x y*, +, *y* squared;

that is, the correct expansion contains terms such as 2*x y*.

Case 4: $(-x)^2 \neq -x^2$

Open parenthesis, negative *x*, close parenthesis, squared, is not equal to the negative of, *x* squared

Instead, $(-x)^2 = x^2$.

Open parenthesis, negative *x*, close parenthesis, squared =, *x* squared
Note carefully where each negative sign appears.

Case 5: $\sqrt{x^2 + y^2} \neq x + y$

The positive square root of the quantity *x* squared + *y* squared, is not equal to *x* + *y*

Case 6: $\frac{a}{x + y} \neq \frac{a}{x} + \frac{a}{y}$

The expression with numerator *a* and denominator *x* + *y*, is not equal to *a* over *x*, +, *a* over *y*

But it is true that $\frac{x + y}{a} = \frac{x}{a} + \frac{y}{a}$.

the expression with numerator *x* + *y*, and denominator *a* =, *x* over *a*, +, *y* over *a*.

2.3 Solving Linear Equations

To **solve an equation** means to find the values of the variables that make the equation true; that is, the values that **satisfy the equation**. Two equations that have the same solutions are called **equivalent equations**. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$, and are false otherwise. The general method for solving an equation is to find successively simpler equivalent equations so that the simplest equivalent equation makes the solutions obvious.

The following two rules are important for producing equivalent equations.

Rule 1: When the same constant is added to or subtracted from both sides of an equation, the equality is preserved and the new equation is equivalent to the original equation.

Rule 2: When both sides of an equation are multiplied or divided by the same nonzero constant, the equality is preserved and the new equation is equivalent to the original equation.

A **linear equation** is an equation involving one or more variables in which each term in the equation is either a constant term or a variable multiplied by a coefficient. None of the variables are multiplied together or raised to a power greater than 1. For example, $2x + 1 = 7x$ and $10x - 9y - z = 3$ 10*x* minus 9*y* minus *z* = 3 are linear equations, but $x + y^2 = 0$ *x*, +, *y* squared, =, 0 and $xz = 3$ are not.

Linear Equations in One Variable

To solve a linear equation in one variable, simplify each side of the equation by combining like terms. Then use the rules for producing simpler equivalent equations.

Example 2.3.1: Solve the equation $11x - 4 - 8x = 2(x + 4) - 2x$

11*x*, minus 4, minus 8*x*, =, 2 times, open parenthesis, *x* + 4, close parenthesis, minus 2*x*

as follows.

Combine like terms to get $3x - 4 = 2x + 8 - 2x$

3*x* minus 4, =, 2*x* + 8 minus 2*x*

Simplify the right side to get $3x - 4 = 8$ 3*x* minus 4, =, 8

Add 4 to both sides to get $3x - 4 + 4 = 8 + 4$ 3*x*, minus 4, + 4, =, 8 + 4

Divide both sides by 3 to get $\frac{3x}{3} = \frac{12}{3}$ 3*x* over 3 = 12 over 3

Simplify to get $x = 4$

You can always check your solution by substituting it into the original equation.

Note that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ 2*x* + 3, =, 2 times, open parenthesis, 7 + *x*, close parenthesis, has no solution, since it is equivalent to the equation 3 = 14, which is false. Also, it is possible that what looks to be a linear equation turns out to be an identity when you try to solve it. For example, $3x - 6 = -3(2 - x)$ 3*x* minus 6, =, negative 3 times, open parenthesis, 2 minus *x*, close parenthesis is true for all values of *x*, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, *x* and *y*, can be written in the form $a x + b y = c$,

where *a*, *b*, and *c* are real numbers and *a* and *b* are not both zero. For example, $3x + 2y = 8$, is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) *x* comma *y* that makes the equation true when the values of *x* and *y* are substituted into the equation. For example, both pairs $(2, 1)$ and $\left(-\frac{2}{3}, 5\right)$ 2 comma 1, and negative 2 over 3 comma 5 are solutions of the equation $3x + 2y = 8$, but $(1, 2)$ 1 comma 2 is not a solution. A linear equation in two variables has infinitely many solutions. If another linear equation in the same variables is given, it may be possible to find a unique solution of both equations. Two equations with the same variables are called a **system of equations**, and the equations in the system are called **simultaneous equations**. To solve a system of two equations means to find an ordered pair of numbers that satisfies ***both*** equations in the system.

There are two basic methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is manipulated to express one variable in terms of the other. Then the expression is substituted in the other equation.

For example, to solve the system of two equations

$4x + 3y = 13$, and

$x + 2y = 2$

you can express x in the second equation in terms of y as $x = 2 - 2y$.

x = 2 minus 2y.

Then substitute $2 - 2y$ 2 minus 2y for x in the first equation to find the value of y.

The value of y can be found as follows.

Substitute for x in the first equation to get $4(2 - 2y) + 3y = 13$

4 times, open parenthesis, 2 minus 2y, close parenthesis, +, 3y, =, 13

Multiply out the first term and get: $8 - 8y + 3y = 13$

8 minus 8*y*, +, 3*y*, =, 13

Subtract 8 from both sides to get $-8y + 3y = 5$

negative 8*y* + 3*y* = 5

Combine like terms to get $-5y = 5$ negative 5*y* = 5

Divide both sides by -5 negative 5 to get $y = -1$. *y* = negative 1.

Then -1 negative 1 can be substituted for *y* in either equation to find the value of *x*. We use the second equation as follows:

Substitute for *y* in the second equation to get $x + 2(-1) = 2$

x +, 2 times negative 1 = 2

That is, $x - 2 = 2$ *x* minus 2 = 2

Add 2 to both sides to get $x = 4$

In the elimination method, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated either by adding the equations together or by subtracting one from the other. In the example above, multiplying both sides of the second equation, $x + 2y = 2$, by 4 yields $4(x + 2y) = 4(2)$,

4 times, open parenthesis, *x* + 2*y*, close parenthesis, =, 4 times 2,

or $4x + 8y = 8$.

Now you have two equations with the same coefficient of *x*.

$4x + 3y = 13$, and

$4x + 8y = 8$

If you subtract the equation $4x + 8y = 8$ from the equation $4x + 3y = 13$, the result is $-5y = 5$. negative 5*y* = 5. Thus, $y = -1$, *y* = negative 1, and substituting -1 negative 1 for y in either of the original equations yields $x = 4$.

By either method, the solution of the system is $x = 4$ and $y = -1$, *y* = negative 1, or $(x, y) = (4, -1)$. the ordered pair *x* comma *y* = the ordered pair 4 comma negative 1.

2.4 Solving Quadratic Equations

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

a x squared + *bx* + *c* = 0,

where a, b, and c are real numbers and $a \neq 0$. *a* is not equal to 0. When such an equation has solutions, they can be found using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

x = the fraction with numerator negative *b* plus or minus the square root of the quantity *b* squared minus 4*a c*, and denominator 2*a*, where the notation $\pm$ plus or minus is shorthand for indicating two solutions, one that uses the plus sign and the other that uses the minus sign.

Example 2.4.1: In the quadratic equation $2x^2 - x - 6 = 0$

2, *x* squared, minus *x*, minus 6 = 0, we have $a = 2$, $b = -1$, and $c = -6$.

a = 2, *b* = negative 1, and *c* = negative 6.

Therefore, the quadratic formula yields

$$
\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)} \\
&= \frac{1 \pm \sqrt{49}}{4} \\
&= \frac{1 \pm 7}{4}
\end{aligned}
$$

x = the fraction with numerator, negative, open parenthesis, negative 1, close parenthesis, plus or minus the square root of the quantity, open parenthesis, negative 1, close parenthesis, squared, minus 4 times 2 times negative 6, and denominator 2 times 2, which is equal to the fraction with numerator 1 plus or minus the square root of 49 and denominator 4, which is equal to the fraction with numerator 1 plus or minus 7, and denominator 4

Hence the two solutions are $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$.

x = the fraction with numerator 1 + 7, and denominator 4, which is equal to 2, and *x* = the fraction with numerator 1 minus 7, and denominator 4, which is equal to negative 3 over 2.

Quadratic equations have at most two real solutions, as in example 2.4.1 above. However, some quadratic equations have only one real solution. For example, the quadratic equation $x^2 + 4x + 4 = 0$ x squared, + 4x, + 4 = 0 has only one solution, which is $x = -2$. x = negative 2. In this case, the expression under the square root symbol in the quadratic formula is equal to 0, and so adding or subtracting 0 yields the same result. Other quadratic equations have no real solutions; for example, $x^2 + x + 5 = 0$. x squared, + x, + 5 = 0. In this case, the expression under the square root symbol is negative, so the entire expression is not a real number.

Some quadratic equations can be solved more quickly by factoring. For example, the quadratic equation $2x^2 - x - 6 = 0$ 2, x squared, minus x, minus 6 = 0 in example 2.4.1 can be factored as $(2x + 3)(x - 2) = 0$.

open parenthesis, 2x + 3, close parenthesis, times, open parenthesis, x minus 2, close parenthesis, = 0.

When a product is equal to 0, at least one of the factors must be equal to 0, so either $2x + 3 = 0$ or $x - 2 = 0$. x minus 2 = 0.

If $2x + 3 = 0$, then $2x = -3$ and $x = -\frac{3}{2}$.

2x = negative 3, and x = negative 3 over 2.

If $x - 2 = 0$, x minus 2 = 0, then $x = 2$.

Thus the solutions are $-\frac{3}{2}$ and 2. negative 3 over 2, and 2.

Example 2.4.2: The quadratic equation $5x^2 + 3x - 2 = 0$

5, x squared, + 3x, minus 2 = 0

can be easily factored as $(5x - 2)(x + 1) = 0$.

open parenthesis, 5*x*, minus 2, close parenthesis, times, open parenthesis, *x* + 1, close parenthesis, = 0.

Therefore, either $5x - 2 = 0$, 5*x*, minus 2 = 0, or $x + 1 = 0$.

If $5x - 2 = 0$, 5*x*, minus 2 = 0, then $x = \frac{2}{5}$. *x* = 2 over 5.

If $x + 1 = 0$, then $x = -1$. *x* = negative 1.

Thus the solutions are $\frac{2}{5}$ and -1. 2 over 5, and negative 1.

2.5 Solving Linear Inequalities

A mathematical statement that uses one of the following four inequality signs is called an **inequality**.

Note: The four inequality signs are given as graphics. Since the meaning of each is given directly after the graphic, a "green font" verbal description of these symbols is not included.

$<$ the less than sign

$>$ the greater than sign

$\leq$ the less than or equal to sign

$\geq$ the greater than or equal to sign

Inequalities can involve variables and are similar to equations, except that the two sides are related by one of the inequality signs instead of the equality sign used in equations. For example, the inequality $4x - 1 \le 7$ 4*x* minus 1, followed by the less than or equal to sign, followed by the number 7 is a linear inequality in one variable, which states that "$4x - 1$ 4*x* minus 1 is less than or equal to 7". To **solve an inequality** means to find the set of all values of the variable that make the inequality true. This set of values is also known as the **solution set** of an inequality. Two inequalities that have the same solution set are called **equivalent inequalities**.

The procedure used to solve a linear inequality is similar to that used to solve a linear equation, which is to simplify the inequality by isolating the variable on one side of the inequality, using the following two rules.

Rule 1: When the same constant is added to or subtracted from both sides of an inequality, the direction of the inequality is preserved and the new inequality is equivalent to the original.

Rule 2: When both sides of the inequality are multiplied or divided by the same nonzero constant, the direction of the inequality is ***preserved if the constant is positive*** but the direction is ***reversed if the constant is negative***. In either case, the new inequality is equivalent to the original.

Example 2.5.1: The inequality $-3x + 5 \le 17$ negative 3*x*, +, 5 is less than or equal to 17 can be solved as follows.

Subtract 5 from both sides to get $-3x \le 12$ negative 3*x* is less than or equal to 12

Divide both sides by -3 negative 3 and reverse the direction of the inequality to get

$\frac{-3x}{-3} \geq \frac{12}{-3}.$

negative 3*x* over negative 3 is greater than or equal to 12 over negative 3.

That is, $x \geq -4.$ *x* is greater than or equal to negative 4.

Therefore, the solution set of

$-3x + 5 \leq 17$ negative 3*x*, +, 5, is less than or equal to 17

consists of all real numbers greater than or equal to $-4.$ negative 4.

Example 2.5.2*:* The inequality $\frac{4x+9}{11} < 5$ the algebraic expression with numerator 4*x* + 9 and denominator 11, is less than 5 can be solved as follows.

Multiply both sides by 11 to get $4x + 9 < 55.$ 4*x* + 9 is less than 55.

Subtract 9 from both sides to get $4x < 46.$ 4*x* is less than 46.

Divide both sides by 4 to get $x < \frac{46}{4}.$ *x* is less than 46 over 4.

That is, $x < 11.5.$ *x* is less than 11.5.

Therefore, the solution set of the inequality $\frac{4x+9}{11} < 5$ the algebraic expression with numerator 4*x* + 9 and denominator 11, is less than 5 consists of all real numbers less than 11.5.

2.6 Functions

An algebraic expression in one variable can be used to define a **function** of that variable. Functions are usually denoted by letters such as *f*, *g*, and *h*. For example, the algebraic expression $3x + 5$ can be used to define a function *f* by

$f(x) = 3x + 5,$ *f* of, *x* = 3*x* + 5,

where $f(x)$ *f* of, *x* is called the value of *f* at *x* and is obtained by substituting the value of *x* in the expression above. For example, if $x = 1$ is substituted in the expression above, the result is $f(1) = 8.$ *f* of, 1 = 8.

It might be helpful to think of a function *f* as a machine that takes an input, which is a value of the variable *x*, and produces the corresponding output, $f(x).$ *f* of, *x*. For any function, each input *x* gives exactly one output $f(x).$ *f* of, *x*. However, more than one value of *x* can give the same output $f(x).$ *f* of, *x*. For example, if *g* is the function defined by $g(x) = x^2 - 2x + 3,$ *g* of, *x* = *x* squared, minus 2*x*, +, 3,

then $g(0) = 3$ and $g(2) = 3.$ *g* of, 0 = 3 and *g* of, 2 = 3.

The **domain** of a function is the set of all permissible inputs, that is, all permissible values of the variable *x*. For the functions *f* and *g* defined above, the domain is the set of all real numbers. Sometimes the domain of the function is given explicitly and is restricted to a specific set of values of *x*. For example, we can define the function *h* by

$h(x) = x^2 - 4,$ **for** $-2 \le x \le 2.$ *h* of, *x* = *x* squared minus 4, for, negative 2 less than or equal to *x*, which is less than or equal to 2.

Without an explicit restriction, the domain is assumed to be the set of all values of *x* for which $f(x)$ *f* of, *x* is a real number.

Example 2.6.1: Let *f* be the function defined by $f(x) = \frac{2x}{x-6}.$

f of, *x* = the algebraic expression with numerator 2*x*, and denominator, *x* minus 6.

In this case, *f* is not defined at *x* = 6, because $\frac{12}{0}$ 12 over 0 is not defined.

Hence, the domain of *f* consists of all real numbers except for 6.

Example 2.6.2: Let *g* be the function defined by $g(x) = x^3 + \sqrt{x+2} - 10.$

g of, *x* = *x* cubed, +, the positive square root of *x* + 2, minus 10.

In this case, $g(x)$ *g* of, *x* is not a real number if $x < -2.$ *x* is less than negative 2. Hence, the domain of *g* consists of all real numbers *x* such that $x \ge -2.$ *x* is greater than or equal to negative 2.

Example 2.6.3: Let *h* be the function defined by $h(x) = |x|,$ *h* of, *x* = the absolute value of *x*, which is the distance between *x* and 0 on the number line (see Chapter 1: Arithmetic, Section 1.5). The domain of *h* is the set of all real numbers. Also, $h(x) = h(-x)$ *h* of, *x* = *h* of, negative *x* for all real numbers *x*, which reflects the property that on the number line the distance between *x* and 0 is the same as the distance between $-x$ and 0. negative *x* and 0.

2.7 Applications

Translating verbal descriptions into algebraic expressions is an essential initial step in solving word problems. Three examples of verbal descriptions and their translations are given below.

Example A: If the square of the number x is multiplied by 3, and then 10 is added to that product, the result can be represented algebraically by $3x^2 + 10$. 3, *x* squared, +, 10.

Example B: If John's present salary s is increased by 14 percent, then his new salary can be represented algebraically by $1.14s$.

Example C: If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then the number of gallons of syrup each of those 4 people will get can be represented algebraically by

$\frac{y-1}{4}$. the expression with numerator *y* minus 1, and denominator 4.

The remainder of this section gives examples of various applications.

Applications Involving Average, Mixture, Rate, and Work Problems

Example 2.7.1: Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to receive on the next exam so that the average (arithmetic mean) score for the 4 exams will be 85 ?

Solution: Let x represent the score on Ellen's next exam. This initial step of assigning a variable to the quantity that is sought is an important beginning to solving the problem. Then in terms of x, the average of the 4 exams is

$\frac{82 + 74 + 90 + x}{4}$, the fraction with numerator 82 + 74 + 90 + x, and denominator 4,

which is supposed to equal 85. Now simplify the expression and set it equal to 85:

$$\frac{82 + 74 + 90 + x}{4} = \frac{246 + x}{4} = 85.$$

the fraction with numerator 82 + 74 + 90 + x, and denominator 4, =, the fraction with numerator 246 + x, and denominator 4, which is equal to 85.

Solving the resulting linear equation for x, you get $246 + x = 340$, and $x = 94$.

Therefore, Ellen will need to attain a score of 94 on the next exam.

Example 2.7.2: A mixture of 12 ounces of vinegar and oil is 40 percent vinegar, where all of the measurements are by weight. How many ounces of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of ounces of oil to be added. Then the total number of ounces of the new mixture will be $12 + x$ and the total number of ounces of vinegar in the new mixture will be $(0.40)(12)$. 0.40 times 12.

Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12+x} = 0.25.$$

the fraction with numerator 0.40 times 12 and denominator 12 + *x*, =, 0.25.

Therefore, $(0.40)(12) = (12 + x)(0.25)$.

0.40 times 12, =, open parenthesis, 12 + *x*, close parenthesis, times 0.25.

Multiplying out gives $4.8 = 3 + 0.25x$, so $1.8 = 0.25x$, and $7.2 = x$.

Thus, 7.2 ounces of oil must be added to produce a new mixture that is 25 percent vinegar.

Example 2.7.3: In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x be the time, in minutes, that it took Dennis to drive the course. The distance d, in miles, is equal to the product of the rate r, in miles per hour, and the time t, in hours; that is,

$d = rt$.

Note that since the rates are given in miles per ***hour***, it is necessary to express the times in hours; for example, 40 minutes equals $\frac{40}{60}$ 40 over 60 of an hour. Thus, the distance traveled by Jeff is the product of his speed and his time, $(51)\left(\frac{40}{60}\right)$ miles,

51 times, open parenthesis, 40 over 60, close parenthesis, miles,

and the distance traveled by Dennis is similarly represented by $(54)\left(\frac{x}{60}\right)$ miles.

54 times, open parenthesis, *x* over 60, close parenthesis, miles.

Since the distances are equal, it follows that $(51)\left(\frac{40}{60}\right) = (54)\left(\frac{x}{60}\right)$.

51, times, open parenthesis, 40 over 60, close parenthesis, =, 54, times, open parenthesis, *x* over 60, close parenthesis.

From this equation it follows that $(51)(40) = 54x$

51 times 40 = 54*x*

and $x = \frac{(51)(40)}{54} \approx 37.8.$

x = the fraction with numerator 51 times 40 and denominator 54, which is approximately 37.8.

Thus, it took Dennis approximately 37.8 minutes to drive the course.

Example 2.7.4: Working alone at its constant rate, machine *A* takes 3 hours to produce a batch of identical computer parts. Working alone at its constant rate, machine *B* takes 2 hours to produce an identical batch of parts. How long will it take the two machines, working simultaneously at their respective constant rates, to produce an identical batch of parts?

Solution: Since machine *A* takes 3 hours to produce a batch, machine *A* can produce $\frac{1}{3}$ one third of the batch in 1 hour. Similarly, machine *B* can produce $\frac{1}{2}$ one half of the batch in 1 hour. If we let *x* represent the number of hours it takes both machines, working simultaneously, to produce the batch, then the two machines will produce $\frac{1}{x}$ 1 over *x* of the job in 1 hour. When the two machines work together, adding their individual production rates, $\frac{1}{3}$ and $\frac{1}{2}$, one third and one half, gives their combined production rate $\frac{1}{x}$. 1 over *x*. Therefore, it follows that $\frac{1}{3} + \frac{1}{2} = \frac{1}{x}$.

one third, +, one half, =, 1 over *x*.

This equation is equivalent to $\frac{2}{6} + \frac{3}{6} = \frac{1}{x}$.

2 over 6, +, 3 over 6, =, 1 over *x*.

So $\frac{5}{6} = \frac{1}{x}$ and $\frac{6}{5} = x$.

5 over 6, =, 1 over *x*, and, 6 over 5, =, *x*.

Thus, working together, the machines will take $\frac{6}{5}$ hours, 6 over 5 hours, or 1 hour 12 minutes, to produce a batch of parts.

Example 2.7.5: At a fruit stand, apples can be purchased for \$0.15 each and pears for \$0.20 each. At these rates, a bag of apples and pears was purchased for \$3.80. If the bag contained 21 pieces of fruit, how many of the pieces were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, the information can be translated into the following system of two equations.

Total Cost Equation: $0.15a + 0.20p = 3.80$

Total Number of Fruit Equation: $a + p = 21$

From the total number of fruit equation, $a = 21 - p$. a = 21 minus p.

Substituting $21 - p$ 21 minus p into the total cost equation for a gives the equation

$0.15(21 - p) + 0.20p = 3.80$

0.15 times, open parenthesis, 21 minus p, close parenthesis, +, 0.20p, =, 3.80

So, $(0.15)(21) - 0.15p + 0.20p = 3.80$,

0.15 times 21, minus, 0.15p, +, 0.20p, =, 3.80,

which is equivalent to $3.15 - 0.15p + 0.20p = 3.80$

3.15, minus, 0.15p, +, 0.20p, =, 3.80.

Therefore $0.05p = 0.65$, and $p = 13$.

Thus, of the 21 pieces of fruit, 13 were pears.

Example 2.7.6: To produce a particular radio model, it costs a manufacturer \$30 per radio, and it is assumed that if 500 radios are produced, all of them will be sold. What must be the selling price per radio to ensure that the profit (revenue from the sales minus the total production cost) on the 500 radios is greater than \$8,200 ?

Solution: If y represents the selling price per radio, then the profit is $500(y - 30)$.

500 times, open parenthesis, y minus 30, close parenthesis.

Therefore, $500(y - 30) > 8,200$.

500 times, open parenthesis, y minus 30, close parenthesis is greater than 8,200.

Multiplying out gives $500y - 15,000 > 8,200$,

500y minus 15,000 is greater than 8,200, which simplifies to $500y > 23{,}200$

500*y* is greater than 23,200

and then to $y > 46.4$. *y* is greater than 46.4. Thus, the selling price must be greater than $46.40 to ensure that the profit is greater than $8,200.

Applications Involving Interest

Some applications involve computing **interest** earned on an investment during a specified time period. The interest can be computed as simple interest or compound interest.

Simple interest is based only on the initial deposit, which serves as the amount on which interest is computed, called the **principal**, for the entire time period. If the amount *P* is invested at a ***simple annual interest rate of r percent***, then the value *V* of the investment at the end of *t* years is given by the formula

$$V = P\left(1 + \frac{rt}{100}\right)$$

V = *P* times, open parenthesis, 1, +, *rt* over 100, close parenthesis,

where *P* and *V* are in dollars.

In the case of **compound interest**, interest is added to the principal at regular time intervals, such as annually, quarterly, and monthly. Each time interest is added to the principal, the interest is said to be compounded. After each compounding, interest is earned on the new principal, which is the sum of the preceding principal and the interest just added. If the amount *P* is invested at an ***annual interest rate of r percent***,

compounded annually, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^{t}.$$

$V = P$ times, open parenthesis, 1, +, r over 100, close parenthesis, to the power t.

If the amount P is invested at an ***annual interest rate of r percent, compounded n times per year***, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100n}\right)^{nt}.$$

$V = P$ times, open parenthesis, 1, +, r over $100n$, close parenthesis, to the power nt.

Example 2.7.7: If \$10,000 is invested at a simple annual interest rate of 6 percent, what is the value of the investment after half a year?

Solution: According to the formula for simple interest, the value of the investment after $\frac{1}{2}$ one half year is

$$\$10,000\left(1 + 0.06\left(\frac{1}{2}\right)\right) = \$10,000(1.03) = \$10,300.$$

\$10,000 times, open parenthesis, 1, +, 0.06 times one half, close parenthesis, =, \$10,000 times 1.03, which is equal to \$10,300.

Example 2.7.8: If an amount P is to be invested at an annual interest rate of 3.5 percent, compounded annually, what should be the value of P so that the value of the investment is $1,000 at the end of 3 years?

Solution: According to the formula for 3.5 percent annual interest, compounded annually, the value of the investment after 3 years is

$$P(1+0.035)^3,$$

P times, open parenthesis, 1 + 0.035, close parenthesis, to the power 3,

and we set it to be equal to $1,000.

$$P(1+0.035)^3 = \$1{,}000.$$

P times, open parenthesis, 1 + 0.035, close parenthesis, to the power 3, =, $1,000.

To find the value of *P*, we divide both sides of the equation by $(1+0.035)^3$.

open parenthesis 1 + 0.035, close parenthesis, to the power 3.

$$P = \frac{\$1{,}000}{(1+0.035)^3} \approx \$901.94.$$

P = $1,000 over, open parenthesis, 1 + 0.035, close parenthesis, to the power 3, which is approximately equal to $901.94.

Thus, an amount of approximately $901.94 should be invested.

Example 2.7.9: A college student expects to earn at least \$1,000 in interest on an initial investment of \$20,000. If the money is invested for one year at interest compounded quarterly, what is the least annual interest rate that would achieve the goal?

Solution: According to the formula for *r* percent annual interest, compounded quarterly, the value of the investment after 1 year is

$$\$20,000\left(1+\frac{r}{400}\right)^4.$$

\$20,000 times, open parenthesis, 1, +, *r* over 400, close parenthesis, to the power 4.

By setting this value greater than or equal to \$21,000 and solving for *r*, we get

$$\$20,000\left(1+\frac{r}{400}\right)^4 \geq \$21,000,$$

\$20,000 times, open parenthesis, 1, +, *r* over 400, close parenthesis, to the power 4, is greater than or equal to \$21,000,

which simplifies to $\left(1+\frac{r}{400}\right)^4 \geq 1.05.$

open parenthesis, 1, +, *r* over 400, close parenthesis, to the power 4, is greater than or equal to 1.05.

Recall that taking the positive fourth root of each side of an inequality preserves the direction of the inequality. (It is also true that taking the positive square root or any other positive root of each side of an inequality preserves the direction of the inequality). Using this fact, we get that taking the positive fourth root of both sides of

$\left(1+\frac{r}{400}\right)^4 \geq 1.05$ open parenthesis, 1, +, *r* over 400, close parenthesis, to the power 4, is greater than or equal to 1.05

yields $1+\frac{r}{400} \geq \sqrt[4]{1.05}$ 1, +, *r* over 400 is greater than or equal to the positive fourth root of 1.05

which simplifies to $r \geq 400\left(\sqrt[4]{1.05}-1\right)$. *r* is greater than or equal to 400 times, open parenthesis, the positive fourth root of 1.05, minus 1, close parenthesis.

To compute the positive fourth root of 1.05, recall that for any number $x \geq 0$,
x greater than or equal to 0,

$\sqrt[4]{x} = \sqrt{\sqrt{x}}$. the positive fourth root of *x*, =, the positive square root of the positive square root of *x*.

This allows us to compute the positive fourth root 1.05 by taking the positive square root of 1.05 and then take the positive square root of the result.

Therefore we can conclude that $400\left(\sqrt[4]{1.05}-1\right) = 400\left(\sqrt{\sqrt{1.05}}-1\right) \approx 4.91$.

400 times, open parenthesis, the positive fourth root of 1.05, minus 1, close parenthesis, =, 400 times, open parenthesis, the positive square root of the positive square root of 1.05, minus 1, close parenthesis, which is approximately 4.91.

Since $r \geq 400\left(\sqrt[4]{1.05}-1\right)$

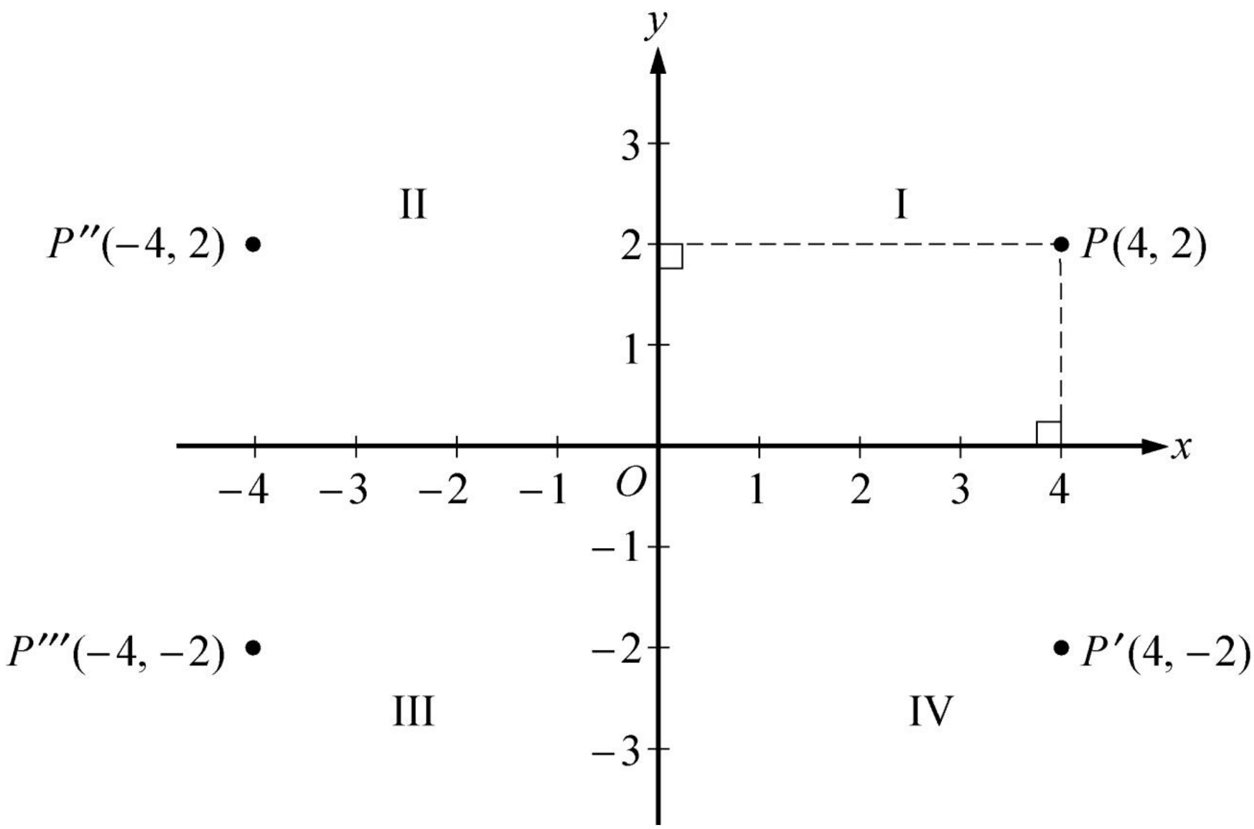
r is greater than or equal to 400 times, open parenthesis, the positive fourth root of 1.05, minus 1, close parenthesis, and $400\left(\sqrt[4]{1.05}-1\right)$

400 times, open parenthesis, the positive fourth root of 1.05, minus 1, close parenthesis

is approximately 4.91, the least annual interest rate is approximately 4.91 percent.

2.8 Coordinate Geometry

Two real number lines that are perpendicular to each other and that intersect at their respective zero points define a **rectangular coordinate system**, often called the ***x y* coordinate system** or ***x y* plane**. The horizontal number line is called the ***x* axis** and the vertical number line is called the ***y* axis**. The point where the two axes intersect is called the **origin**, denoted by *O*. The positive half of the *x* axis is to the right of the origin, and the positive half of the *y* axis is above the origin. The two axes divide the plane into four regions called **quadrants**. The four quadrants are labeled I, II, III, and IV, 1, 2, 3, **and** 4, as shown in Algebra Figure 1 below.



Algebra Figure 1

Begin skippable part of description of Algebra Figure 1.

Quadrant I 1 is the region of the *x y* plane that is above the *x* axis and to the right of the *y* axis. Quadrant II 2 is the region that is above the *x* axis and to the left of the *y* axis. Quadrant III 3 is the region that is below the *x* axis and to the left of the *y* axis. Quadrant IV 4 is the region that is below the *x* axis and to the right of the *y* axis.

There are equally spaced tick marks along each of the axes. Along the *x* axis, to the right of the origin the tick marks are labeled 1, 2, 3, and 4; and to the left of the origin the tick marks are labeled $-1, -2, -3,$ and -4. negative 1, negative 2, negative 3, and negative

4. Along the *y* axis, above the origin the tick marks are labeled 1, 2, and 3; and below the origin the tick marks are labeled $-1, -2$, and -3. negative 1, negative 2, and negative 3.

End skippable part of figure description.

Each point in the *x y* plane can be identified with an ordered pair (x, y) *x* comma *y* of real numbers. The first number in the ordered pair is called the ***x* coordinate**, and the second number is called the ***y* coordinate**. A point with coordinates (x, y) *x* comma *y* is located $|x|$ units the absolute value of *x* units to the right of the *y* axis if *x* is positive, or to the left of the *y* axis if *x* is negative. Also, the point is located $|y|$ units the absolute value of *y* units above the *x* axis if *y* is positive, or below the *x* axis if *y* is negative. If *x* = 0, the point lies on the *y* axis, and if *y* = 0 the point lies on the *x* axis. The origin has coordinates $(0, 0)$. 0 comma 0. Unless otherwise noted, the units used on the *x* axis and the *y* axis are the same.

A point *P* with coordinates (x, y) *x* comma *y* is denoted by $P(x, y)$. *P*, open parenthesis *x* comma *y*, close parenthesis.

In Algebra Figure 1 above, the point $P(4, 2)$ *P* with coordinates 4 comma 2 is 4 units to the right of the *y* axis and 2 units above the *x* axis, the point $P'(4, -2)$ *P* prime with coordinates 4 comma negative 2 is 4 units to the right of the *y* axis and 2 units below the *x* axis, the point $P''(-4, 2)$ *P* double prime with coordinates negative 4 comma 2 is 4 units to the left of the *y* axis and 2 units above the *x* axis, and the point $P'''(-4, -2)$ *P* triple prime with coordinates negative 4 comma negative 2 is 4 units to the left of the *y* axis and 2 units below the *x* axis.

Note that the three points $P'(4, -2)$, $P''(-4, 2)$, and $P'''(-4, -2)$

P prime with coordinates 4 comma negative 2, *P* double prime with coordinates negative 4 comma 2, and *P* triple prime with coordinates negative 4 comma negative 2

have the same coordinates as *P* except for the signs. These points are geometrically related to *P* as follows.

P' *P* prime is the **reflection of *P* about the *x* axis**, or P' *P* prime and *P* are **symmetric about the *x* axis**.

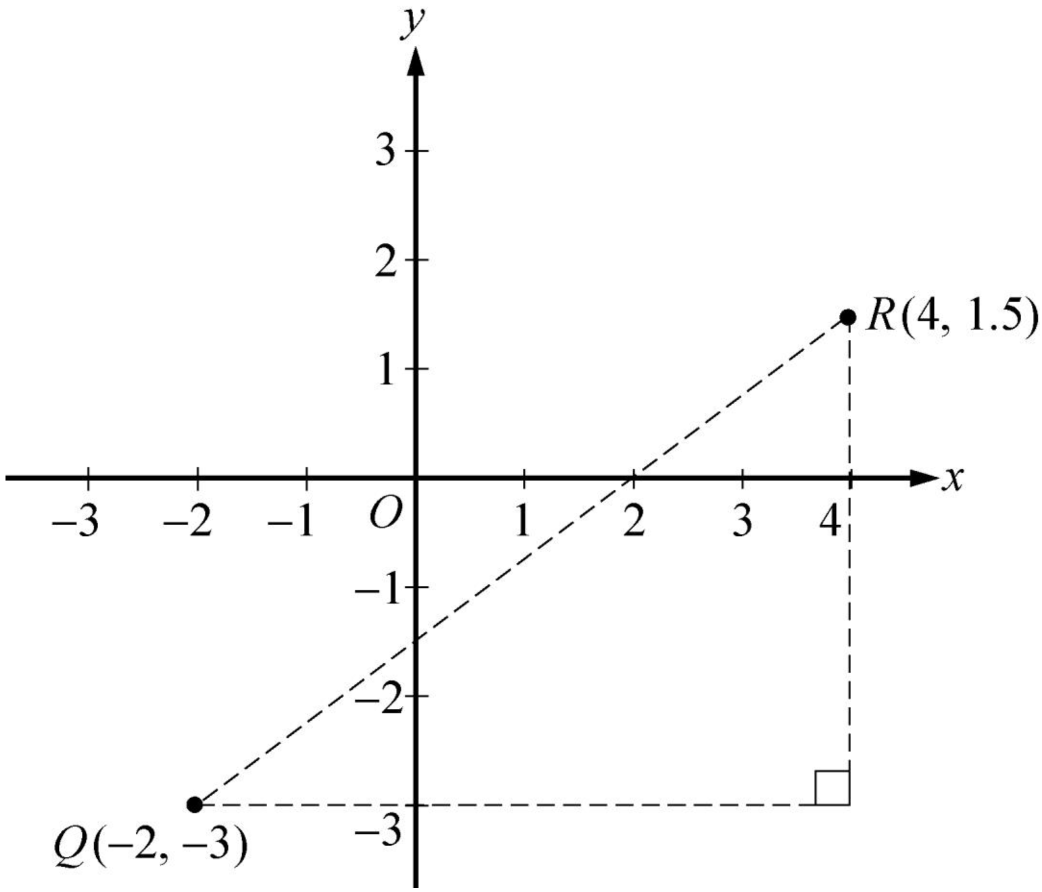
P'' *P* double prime is the **reflection of *P* about the *y* axis**, or P'' *P* double prime and *P* are **symmetric about the *y* axis**.

P''' *P* triple prime is the **reflection of *P* about the origin**, or P''' *P* triple prime and *P* are **symmetric about the origin**.

The distance between two points in the *x y* plane can be found by using the Pythagorean theorem. For example, the distance between the two points $Q(-2, -3)$ and $R(4, 1.5)$

Q with coordinates negative 2 comma negative 3, and *R* with coordinates 4 comma 1.5

in Algebra Figure 2 below is the length of line segment *QR*. To find this length, construct a right triangle with hypotenuse *QR* by drawing a vertical line segment downward from *R* and a horizontal line segment rightward from *Q* until these two line segments intersect at the point with coordinates $(4, -3)$ 4 comma negative 3 forming a right angle, as shown in Algebra Figure 2. Then note that the horizontal side of the triangle has length $4 - (-2) = 6$ 4 minus negative 2 = 6 and the vertical side of the triangle has length $1.5 - (-3) = 4.5.$ 1.5 minus negative 3, =, 4.5.



Algebra Figure 2

Since line segment *QR* is the hypotenuse of the triangle, you can apply the Pythagorean theorem:

$$QR = \sqrt{6^2 + 4.5^2} = \sqrt{56.25} = 7.5.$$

QR = the positive square root of the quantity 6 squared + 4.5 squared, which is equal to the positive square root of 56.25, or 7.5.

(For a discussion of right triangles and the Pythagorean theorem, see Chapter 3: Geometry, Section 3.3).

Equations in two variables can be represented as graphs in the coordinate plane. In the *x y* plane, the **graph of an equation** in the variables *x* and *y* is the set of all points whose ordered pairs (x, y) *x* comma *y* satisfy the equation.

The graph of a linear equation of the form $y = mx + b$ is a straight line in the *x y* plane, where m is called the **slope** of the line and b is called the ***y* intercept**.

The ***x* intercepts** of a graph are the *x* values of the points at which the graph intersects the *x* axis. Similarly, the ***y* intercepts** of a graph are the *y* values of the points at which the graph intersects the *y* axis.

The slope of a line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$, *Q* with coordinates *x* sub 1 comma *y* sub 1, and *R* with coordinates *x* sub 2 comma *y* sub 2,

where $x_1 \neq x_2$, *x* sub 1 is not equal to *x* sub 2, is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

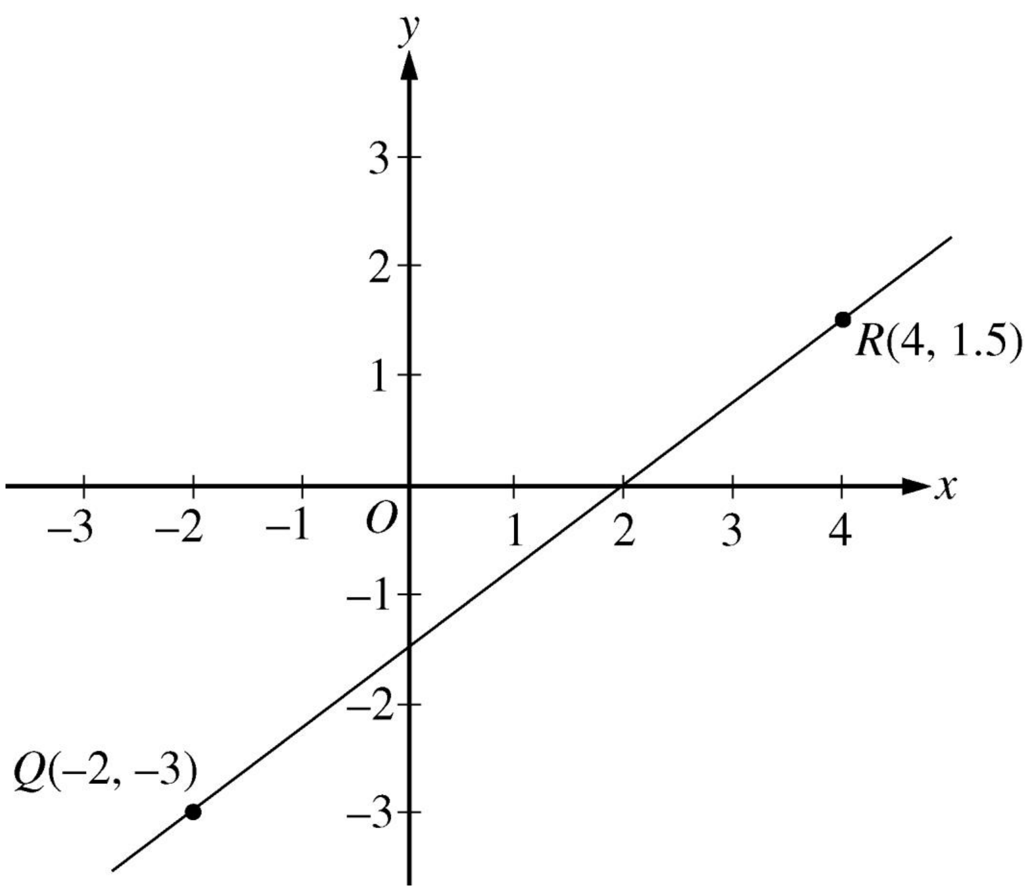
the fraction with numerator *y* sub 2 minus *y* sub 1, and denominator *x* sub 2 minus *x* sub 1.

This ratio is often called "rise over run", where **rise** is the change in *y* when moving from *Q* to *R* and **run** is the change in *x* when moving from *Q* to *R*. A horizontal line has a slope of 0, since the rise is 0 for any two points on the line. So the equation of every horizontal line has the form $y = b$, where b is the *y* intercept. The slope of a vertical line is not defined, since the run is 0. The equation of every vertical line has the form $x = a$, where a is the *x* intercept.

Two lines are **parallel** if their slopes are equal. Two lines are **perpendicular** if their slopes are negative reciprocals of each other. For example, the line with equation $y = 2x + 5$ is perpendicular to the line with equation $y = -\frac{1}{2}x + 9$.

y = negative one half *x*, + 9.

Example 2.8.1: Algebra Figure 3 below shows the graph of the line through points $Q(-2, -3)$ and $R(4, 1.5)$ *Q* with coordinates negative 2 comma negative 3, and *R* with coordinates 4 comma 1.5 in the *x y* plane.



Algebra Figure 3

Begin skippable part of description of Algebra Figure 3.

Point $Q(-2, -3)$ *Q* with coordinates negative 2 comma negative 3 is 2 units to the left, and 3 units below the origin. Point $R(4, 1.5)$ *R* with coordinates 4 comma 1.5 is 4 units to the right, and 1.5 units above the origin. Line *QR* crosses the *y* axis about halfway between -1 and -2, negative 1 and negative 2, and crosses the *x* axis at 2.

End skippable part of figure description.

In Algebra Figure 3 above, the slope of the line passing through the points

$Q(-2, -3)$ and $R(4, 1.5)$ is $\frac{1.5 - (-3)}{4 - (-2)} = \frac{4.5}{6} = 0.75.$

Q with coordinates negative 2 comma negative 3, and *R* with coordinates 4 comma 1.5 is the fraction with numerator 1.5 minus negative 3, and denominator 4 minus negative 2, which is equal to the fraction 4.5 over 6, or 0.75.

Line *QR* appears to intersect the *y* axis close to the point $(0, -1.5)$, with coordinates 0 comma negative 1.5, so the *y* intercept of the line must be close to $-1.5.$ negative 1.5. To get the exact value of the *y* intercept, substitute the coordinates of any point on the line into the equation $y = 0.75x + b$, and solve it for *b*.

For example, if you pick the point $Q(-2, -3)$, *Q* with coordinates negative 2 comma negative 3, and substitute its coordinates into the equation you get

$-3 = (0.75)(-2) + b.$ negative 3 = 0.75 times negative 2, +, *b*.

Then adding $(0.75)(2)$ 0.75 times 2 to both sides of the equation yields

$b = -3 + (0.75)(2)$, or $b = -1.5.$

b = negative 3, +, 0.75 times 2, or *b* = negative 1.5.

Therefore, the equation of line *QR* is $y = 0.75x - 1.5.$

y = 0.75*x*, minus, 1.5.

You can see from the graph in Algebra Figure 3 that the *x* intercept of line *QR* is 2, since *QR* passes through the point $(2, 0).$ 2 comma 0. More generally, you can find the *x* intercept of a line by setting *y* = 0 in an equation of the line and solving it for *x*. So you can find the *x* intercept of line *QR* by setting *y* = 0 in the equation $y = 0.75x - 1.5$ *y* = 0.75*x*, minus, 1.5 and solving it for *x* as follows.

Setting *y* = 0 in the equation $y = 0.75x - 1.5$ *y* = 0.75*x*, minus, 1.5 gives the equation $0 = 0.75x - 1.5.$ 0 = 0.75*x*, minus, 1.5. Then adding 1.5 to both sides yields 1.5 = 0.75*x*. Finally, dividing both sides by 0.75 yields $x = \frac{1.5}{0.75} = 2.$ *x* = 1.5 over 0.75, or 2.

Graphs of linear equations can be used to illustrate solutions of systems of linear equations and inequalities, as can be seen in examples 2.8.2 and 2.8.3.

Example 2.8.2: Consider the system of two linear equations in two variables:

4*x* + 3*y* = 13, and

x + 2*y* = 2

(Note that this system was solved by substitution, and by elimination in Algebra Section 2.3.)

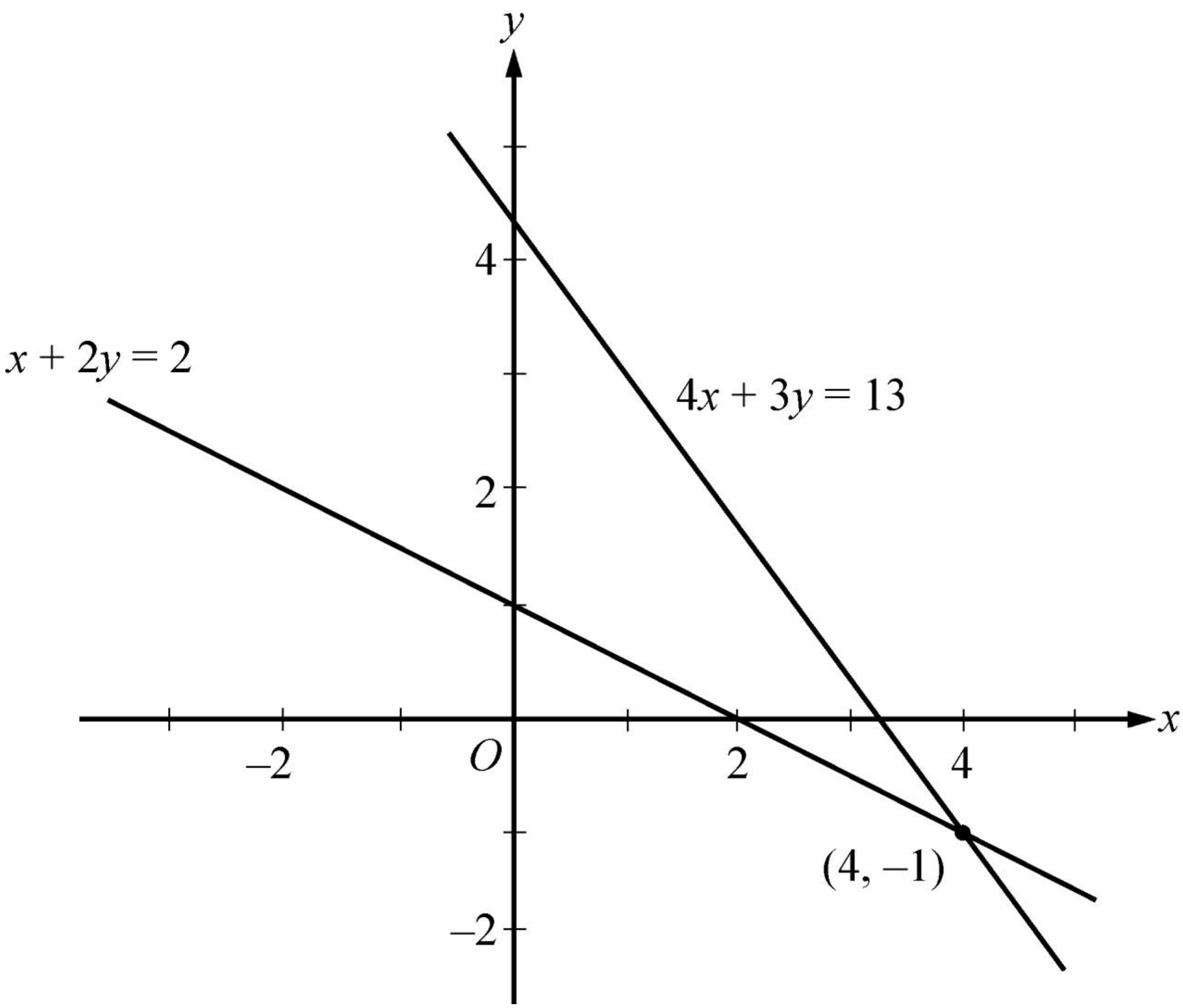
Solving each equation for *y* in terms of *x* yields

$$y = -\frac{4}{3}x + \frac{13}{3}$$

$$y = -\frac{1}{2}x + 1$$

y = negative four thirds *x*, +, 13 over 3, and, *y* = negative one half *x*, +, 1

Algebra Figure 4 below shows the graphs of the two equations in the *x y* plane. The solution of the system of equations is the point at which the two graphs intersect, which is $(4, -1)$. 4 comma negative 1.



Algebra Figure 4

Begin skippable part of description of Algebra Figure 4.

The graphs of both equations are lines that slant downward as they go from left to right. The graph of the equation $4x + 3y = 13$ crosses the y axis at a number that is between 4 and 5, and crosses the x axis at a number that is between 3 and 4. The graph of the equation $x + 2y = 2$ intersects the y axis at 1 and the x axis at 2. The two graphs intersect at the point $(4, -1)$, 4 comma negative 1, which is in the fourth quadrant.

End skippable part of figure description

Example 2.8.3: Consider the following system of two linear inequalities.

$$x - 3y \geq -6, \text{ and}$$
$$2x + y \geq -1$$

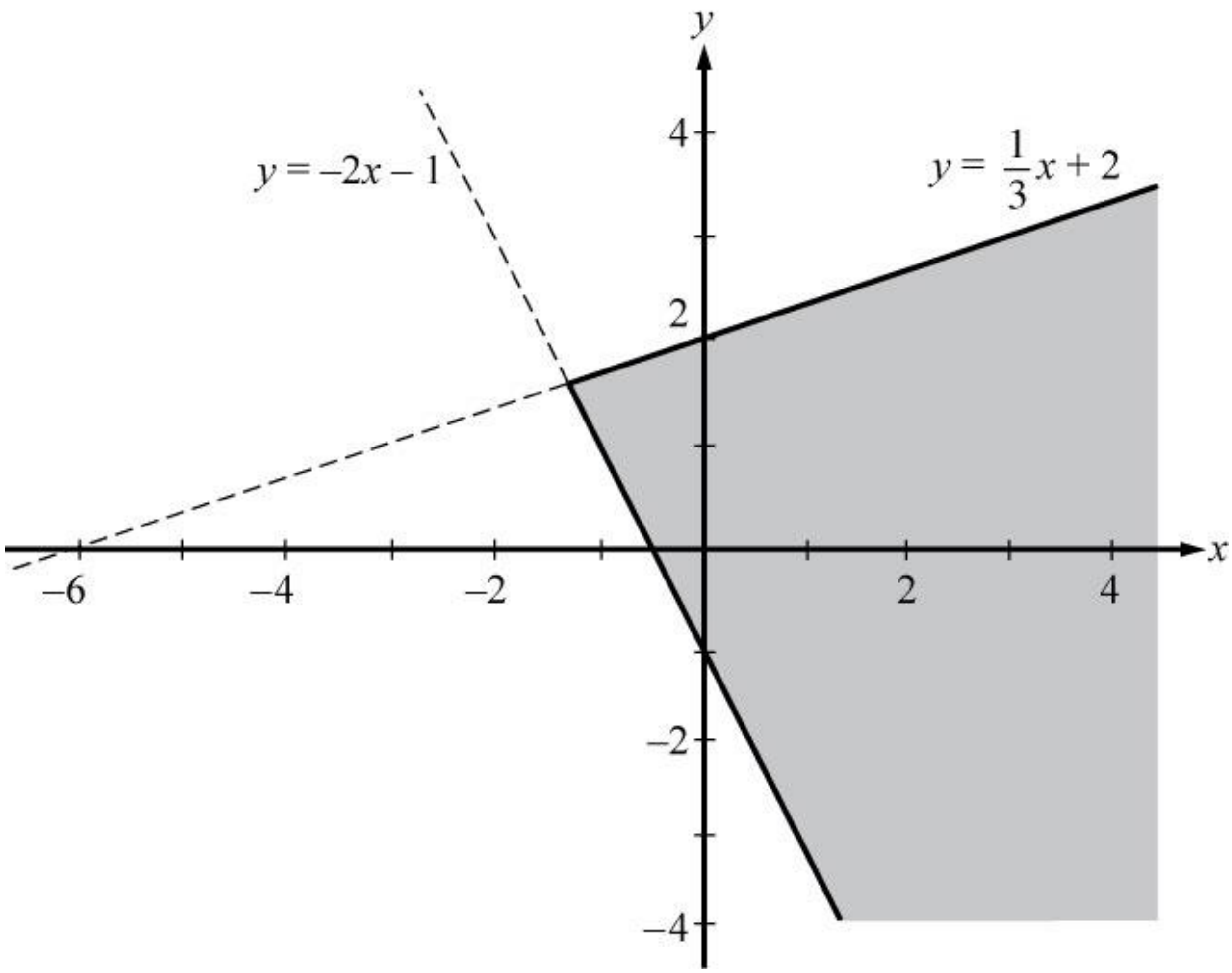
x minus 3*y* is greater than or equal to negative 6, and, 2*x* + *y* is greater than or equal to negative 1

Solving each inequality for *y* in terms of *x* yields

$$y \leq \frac{1}{3}x + 2, \text{ and}$$
$$y \geq -2x - 1$$

y is less than or equal to one third *x*, +, 2, and, *y* is greater than or equal to negative 2*x*, minus 1

Each point (x, y) *x* comma *y* that satisfies the first inequality, $y \leq \frac{1}{3}x + 2$, *y* is less than or equal to one third *x*, +, 2, is either on the line $y = \frac{1}{3}x + 2$ *y* = one third *x*, +, 2 or below the line because the *y* coordinate is either equal to or less than $\frac{1}{3}x + 2$. one third *x*, +, 2. Therefore, the graph of $y \leq \frac{1}{3}x + 2$ *y* is less than or equal to one third *x*, +, 2 consists of the line $y = \frac{1}{3}x + 2$ *y* = one third *x*, +, 2 and the entire region below it. Similarly, the graph of $y \geq -2x - 1$ *y* is greater than or equal to negative 2*x*, minus 1 consists of the line $y = -2x - 1$ *y* = negative 2*x*, minus 1 and the entire region above it. Thus, the solution set of the system of inequalities consists of all of the points that lie in the shaded region shown in Algebra Figure 5 below, which is the intersection of the two regions described.



Algebra Figure 5

Begin skippable part of description of Algebra Figure 5.

The graph of the equation $y = \frac{1}{3}x + 2$ y = one third x, +, 2 is a line that slants upward as it goes from left to right, crossing the *x* axis at -6 negative 6 and the *y* axis at 2. The graph of the equation $y = -2x - 1$ y = negative 2x, minus 1 slants downward as it goes from left to right, crossing the *x* axis between negative 1 and 0 and the *y* axis at -1. negative 1. The two lines intersect in the second quadrant. The region below the graph of $y = \frac{1}{3}x + 2$ y = one third x, +, 2 and to the right of the graph of $y = -2x - 1$ y = negative 2x, minus 1 is shaded.

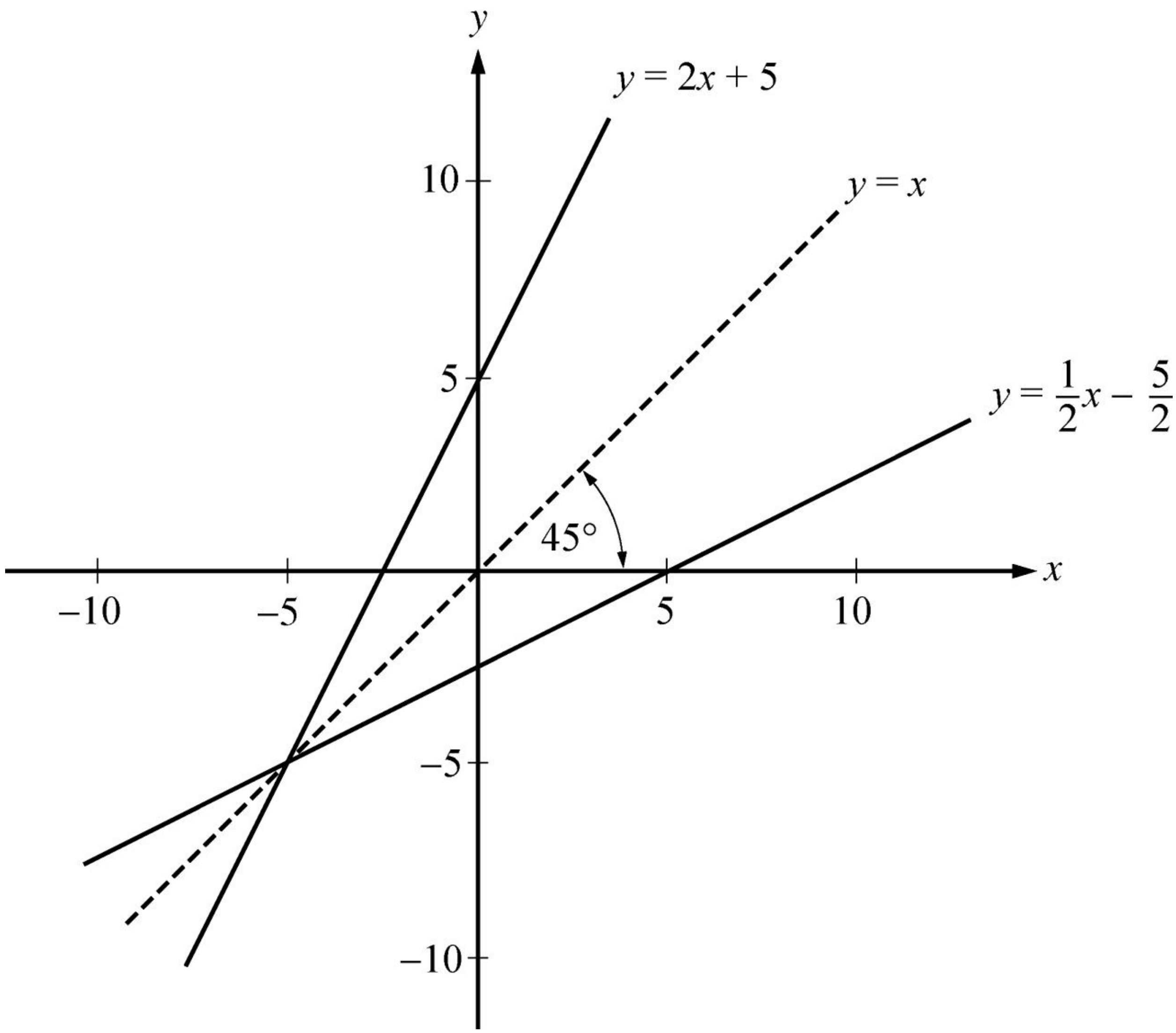
End skippable part of figure description.

Symmetry with respect to the x axis, the y axis, and the origin is mentioned earlier in this section. Another important symmetry is symmetry with respect to the line with equation $y = x$. The line $y = x$ passes through the origin, has a slope of 1, and makes a 45 degree angle with each axis. For any point with coordinates (a, b), a comma b, the point with interchanged coordinates (b, a) b comma a is the reflection of (a, b) a comma b about the line $y = x$; that is, (a, b) and (b, a) a comma b, and, b comma a are symmetric about the line $y = x$. It follows that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line $y = x$.

Example 2.8.4: Consider the line whose equation is $y = 2x + 5$. Interchanging x and y in the equation yields $x = 2y + 5$. Solving this equation for y yields $y = \frac{1}{2}x - \frac{5}{2}$.

y = one half x, minus 5 halves.

The line $y = 2x + 5$ and its reflection $y = \frac{1}{2}x - \frac{5}{2}$ y = one half x, minus 5 halves are graphed in Algebra Figure 6 below.



Algebra Figure 6

Begin skippable part of description of Algebra Figure 6.

The figure shows the lines $y = 2x + 5$ and $y = \frac{1}{2}x - \frac{5}{2}$ *y* = one half *x*, minus 5 halves and the dashed line $y = x$ between them. The three lines intersect at a point in the third quadrant. When the line $y = 2x + 5$ is flipped over the line $y = x$, the result is the line

$y = \frac{1}{2}x - \frac{5}{2}$. *y* = one half *x*, minus 5 halves. The result of this flipping on the intercepts of the lines is that the *y* intercept of *y* = 2*x* + 5 and the *x* intercept of

$y = \frac{1}{2}x - \frac{5}{2}$ *y* = one half *x*, minus 5 halves are equal; and the *x* intercept of

y = 2*x* + 5 and the *y* intercept of the line $y = \frac{1}{2}x - \frac{5}{2}$ *y* = one half *x*, minus 5 halves

are equal.

End skippable part of figure description.

The line *y* = *x* is a **line of symmetry** for the graphs of *y* = 2*x* + 5 and $y = \frac{1}{2}x - \frac{5}{2}$.

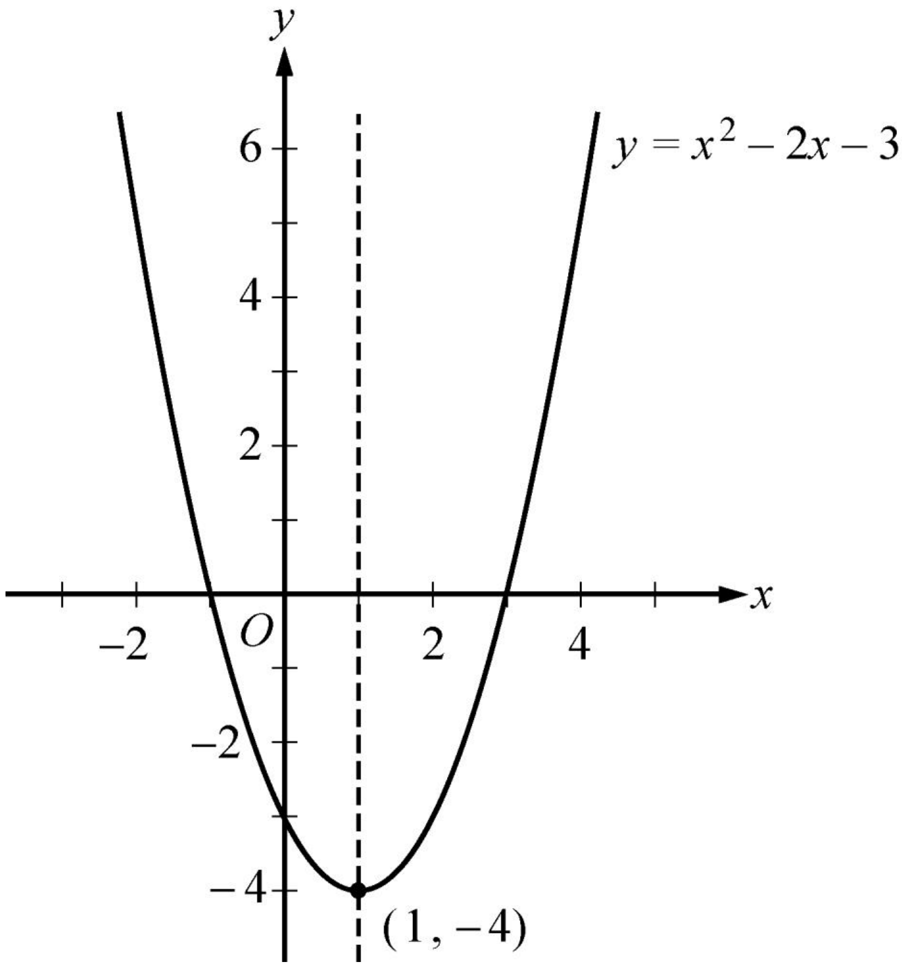
y = one half *x*, minus 5 halves.

The graph of a quadratic equation of the form $y = ax^2 + bx + c$,

y = *a x* squared, +, *bx*, +, *c*, where *a*, *b*, and *c* are constants and $a \neq 0$, *a* is not equal to 0, is a **parabola**. The *x* intercepts of the parabola are the solutions of the equation

$ax^2 + bx + c = 0$. *a x* squared, +, *bx*, +, *c* = 0. If *a* is positive, the parabola opens upward and the **vertex** is its lowest point. If *a* is negative, the parabola opens downward and the vertex is its highest point. Every parabola is symmetric with itself about the vertical line that passes through its vertex. In particular, the two *x* intercepts are equidistant from this line of symmetry.

Example 2.8.5: The equation $y = x^2 - 2x - 3$ *y* = *x* squared, minus 2*x*, minus 3 has the graph shown in Algebra Figure 7 below.



Algebra Figure 7

Begin skippable part of description of Algebra Figure 7.

The figure shows the parabola, which is an upward facing, U shaped curve, and the vertical dashed line $x = 1$.

End skippable part of figure description.

The graph indicates that the x intercepts of the parabola are -1 and 3. negative 1 and 3. The values of the x intercepts can be confirmed by solving the quadratic equation

$x^2 - 2x - 3 = 0$ *x* squared, minus 2*x*, minus 3 = 0 to get $x = -1$ and $x = 3$.

x = negative 1 and *x* = 3. The point $(1, -4)$ 1 comma negative 4 is the vertex of the parabola, and the line *x* = 1 is its line of symmetry. The *y* intercept is the *y* coordinate of the point on the parabola at which *x* = 0, which is $y = 0^2 - 2(0) - 3 = -3$.

y = 0 squared, minus 2 times 0, minus 3 = negative 3.

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$

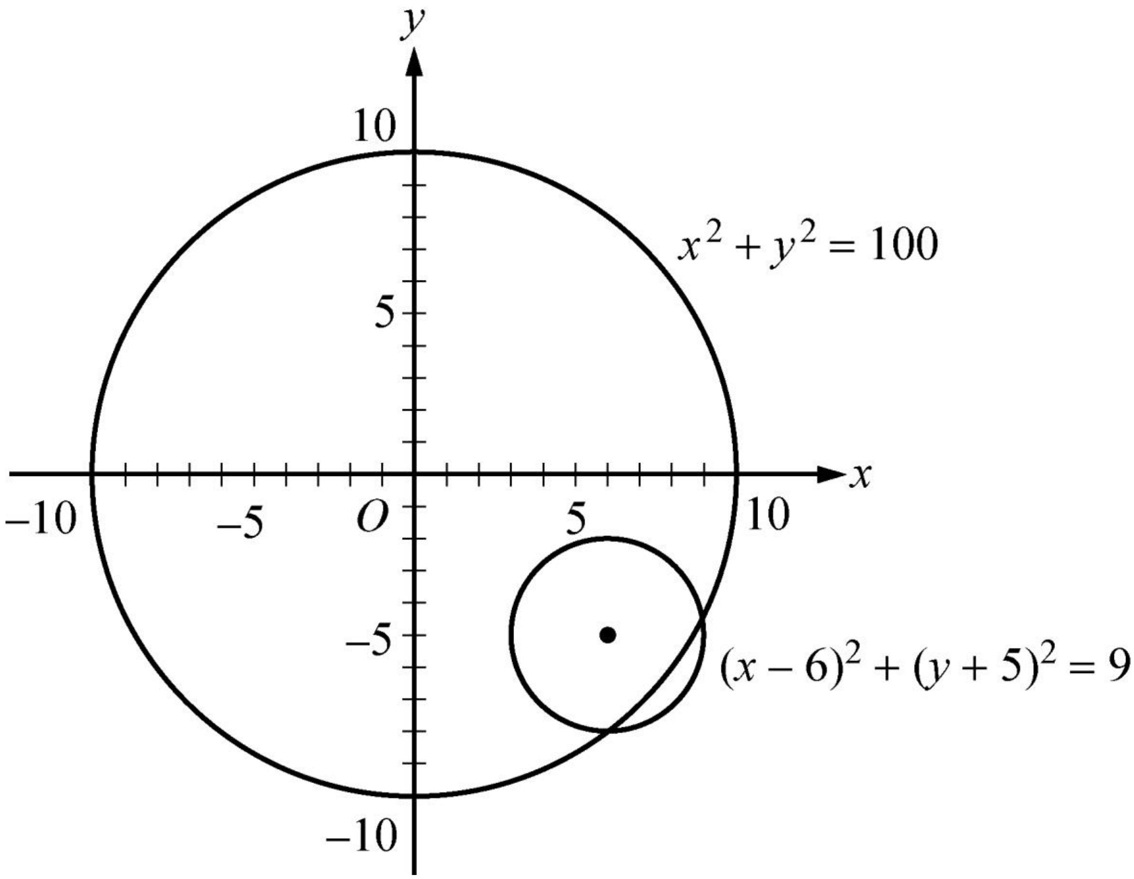
open parenthesis, *x* minus *a*, close parenthesis, squared, +, open parenthesis, *y* minus *b*, close parenthesis, squared, =, *r* squared

is a **circle** with its center at the point (a, b) *a* comma *b* and with radius *r*.

Example 2.8.6: Algebra Figure 8 below shows the graph of two circles in the *x y* plane. The larger of the two circles is centered at the origin and has radius 10, so its equation is $x^2 + y^2 = 100$. *x* squared, +, *y* squared, =, 100. The smaller of the two circles has center $(6, -5)$ 6 comma negative 5 and radius 3, so its equation is

$(x - 6)^2 + (y + 5)^2 = 9$.

open parenthesis, *x* minus 6, close parenthesis, squared, +, open parenthesis, *y* + 5, close parenthesis, squared =, 9.



Algebra Figure 8

Begin skippable part of description of Algebra Figure 8.

The center of the smaller circle is in the fourth quadrant and lies inside the large circle. The two circles intersect at two points, both of which are in the fourth quadrant.

End skippable part of figure description.

2.9 Graphs of Functions

The coordinate plane can be used for graphing functions. To graph a function in the *x y* plane, you represent each input *x* and its corresponding output $f(x)$ *f* of, *x* as a point (x, y), *x* comma *y*, where $y = f(x)$. *y* = *f* of, *x*. In other words, you use the *x* axis for the input and the *y* axis for the output.

Below are several examples of graphs of elementary functions.

Example 2.9.1: Consider the linear function defined by $f(x) = -\frac{1}{2}x + 1$.

f of, *x* = negative one half *x*, +, 1.

Its graph in the *x y* plane is the line with the linear equation $y = -\frac{1}{2}x + 1$.

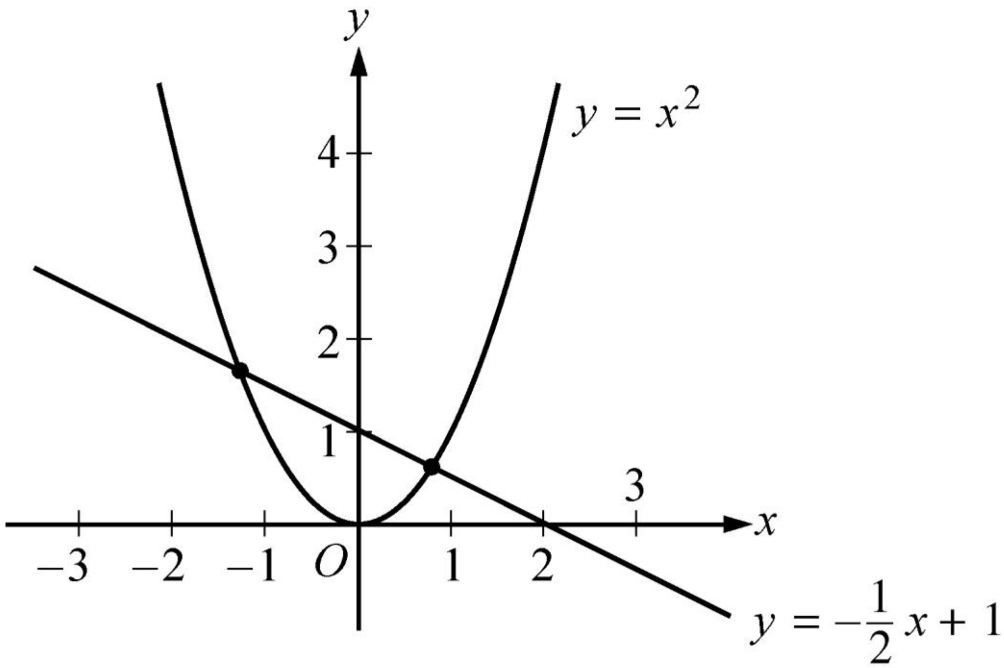
y = negative one half *x*, +, 1.

Example 2.9.2: Consider the quadratic function defined by $g(x) = x^2$.

g of, *x* = *x* squared.

The graph of *g* is the parabola with the quadratic equation $y = x^2$. *y* = *x* squared.

The graph of both the linear equation $y = -\frac{1}{2}x + 1$ *y* = negative one half *x*, +, 1 and the quadratic equation $y = x^2$ *y* = *x* squared are shown in Algebra Figure 9 below.



Algebra Figure 9

Begin skippable part of description of Algebra Figure 9.

The graph of the linear equation $y = -\frac{1}{2}x + 1$ *y* = negative one half *x*, +, 1 is a line that slants downward as it goes from left to right, intersecting the *y* axis at 1, and the *x* axis at 2. The graph of the quadratic equation $y = x^2$ *y* = *x* squared is an upward facing parabola, whose vertex is at the origin.

End skippable part of figure description.

Note that the graphs *f* and *g* in Algebra Figure 9 above intersect at two points. These are the points at which $g(x) = f(x)$. *g* of, *x* = *f* of, *x*. We can find these points algebraically as follows.

Set $g(x) = f(x)$ *g* of, *x* = *f* of, *x* and get $x^2 = -\frac{1}{2}x + 1$, *x* squared, =, negative one half *x*, +, 1, which is equivalent to $x^2 + \frac{1}{2}x - 1 = 0$; or $2x^2 + x - 2 = 0$.

x squared, +, one half *x*, minus, 1, =, 0; or 2, *x* squared, +, *x*, minus 2, =, 0.

Then solve the equation $2x^2 + x - 2 = 0$ 2, *x* squared, +, *x*, minus 2, =, 0 for *x* using the quadratic formula getting $x = \frac{-1 \pm \sqrt{1 + 16}}{4}$, *x* = the fraction with numerator negative 1 plus or minus the square root of the quantity 1 + 16, and denominator 4, which represents the *x* coordinates of the two solutions

$$x = \frac{-1 + \sqrt{17}}{4} \approx 0.78 \quad \text{and} \quad x = \frac{-1 - \sqrt{17}}{4} \approx -1.28.$$

x = the fraction with numerator negative 1 plus the positive square root of 17, and denominator 4, which is approximately 0.78, and *x* = the fraction with numerator negative 1 minus the positive square root of 17, and denominator 4, which is approximately negative 1.28.

With these input values, the corresponding *y* coordinates can be found using either *f* or *g*:

$$g\left(\frac{-1 + \sqrt{17}}{4}\right) = \left(\frac{-1 + \sqrt{17}}{4}\right)^2 \approx 0.61 \quad \text{and}$$
$$g\left(\frac{-1 - \sqrt{17}}{4}\right) = \left(\frac{-1 - \sqrt{17}}{4}\right)^2 \approx 1.64.$$

g of, the fraction with numerator negative 1 + the positive square root of 17 and denominator 4, =, open parenthesis, the fraction with numerator negative 1 + the positive square root of 17 and denominator 4, close parenthesis, squared, which is approximately 0.61, and
g of, the fraction with numerator negative 1 minus the positive square root of 17 and denominator 4, =, open parenthesis, the fraction with numerator negative 1 minus the positive square root of 17 and denominator 4, close parenthesis, squared, which is approximately 1.64.

Thus, the two intersection points can be approximated by

$(0.78, 0.61)$ and $(-1.28, 1.64)$.

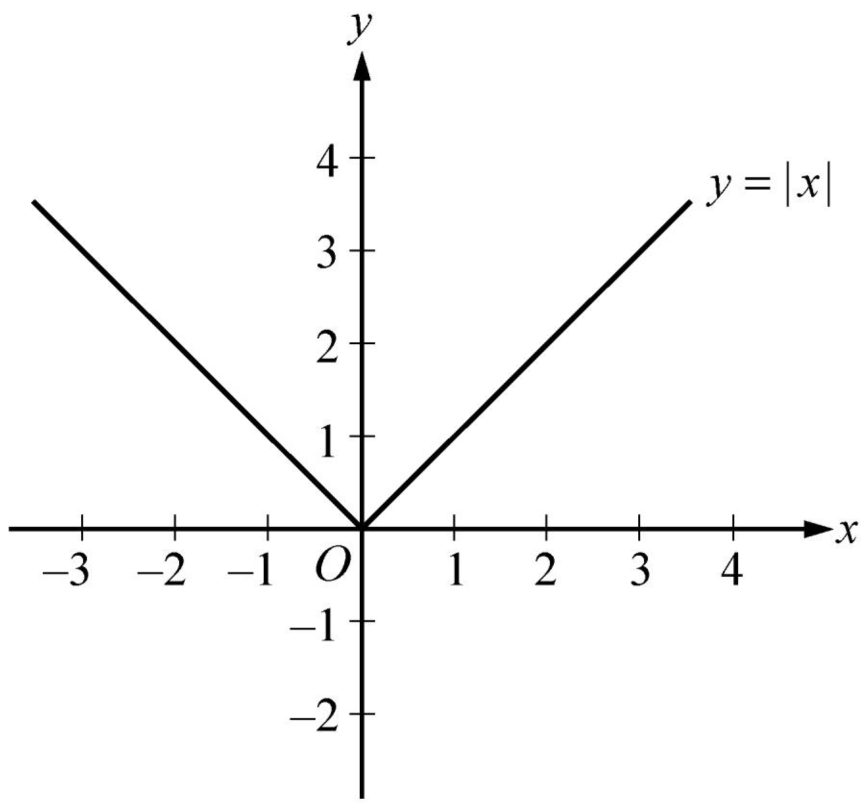
the point 0.78 comma 0.61 and the point negative 1.28 comma 1.64.

Example 2.9.3: Consider the absolute value function defined by $h(x) = |x|$. *h* of, *x* = the absolute value of *x*. By using the definition of absolute value (see Chapter 1: Arithmetic, Section 1.5), *h* can be expressed as a **piecewise defined** function:

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

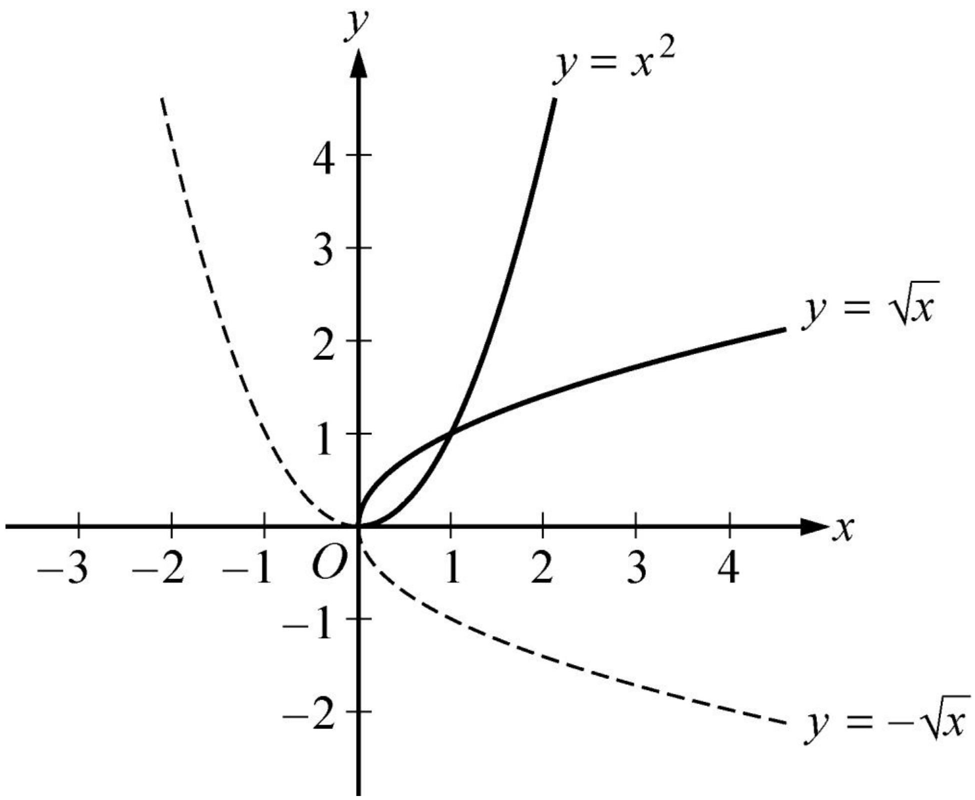
h of, *x* = *x*, for *x* greater than or equal to 0, and *h* of, *x* = negative *x*, for *x* less than 0

The graph of this function is V shaped and consists of two linear pieces, $y = x$ and $y = -x$, *y* = *x* and *y* = negative *x*, joined at the origin, as shown in Algebra Figure 10 below.



Algebra Figure 10

Example 2.9.4: This example is based on Algebra Figure 11 below, which is the graph of 2 parabolas.



Algebra Figure 11

One of the parabolas is the upward facing parabola $y = x^2$ *y* = *x* squared and the other looks like the parabola $y = x^2$, *y* = *x* squared, but instead of facing upward, it faces to the right. The vertex of both parabolas is at the origin.

Begin skippable part of description of Algebra Figure 11.

The half of the upward facing parabola to the right of the *y* axis is a solid curve, and the half to the left of the *y* axis is a dashed curve. The half of the right facing parabola above the *x* axis is a solid curve and the half below the *x* axis is a dashed curve.

End skippable part of figure description.

Consider the positive square root function defined by $j(x) = \sqrt{x}$ for $x \geq 0$.

j of, *x* = the positive square root of *x*, for *x* greater than or equal to 0.

The graph of this function is the upper half of the right facing parabola in Algebra Figure 11; that is, the solid part of the parabola, the part above the *x* axis. Also consider the negative square root function defined by $k(x) = -\sqrt{x}$ for $x \geq 0$.

k of, *x* = the negative square root of *x*, for *x* greater than or equal to 0. The graph of this function is the bottom half of the right facing parabola; that is, the dashed part of the parabola, the part below the *x* axis.

The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ *y* = the positive square root of *x* and *y* = the negative square root of *x* are halves of a parabola because they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ *y* = *x* squared about the line *y* = *x*. This follows from squaring both sides of the two square root equations to get $y^2 = x$ *y* squared = *x* and then interchanging *x* and *y* to get $y = x^2$. *y* = *x* squared.

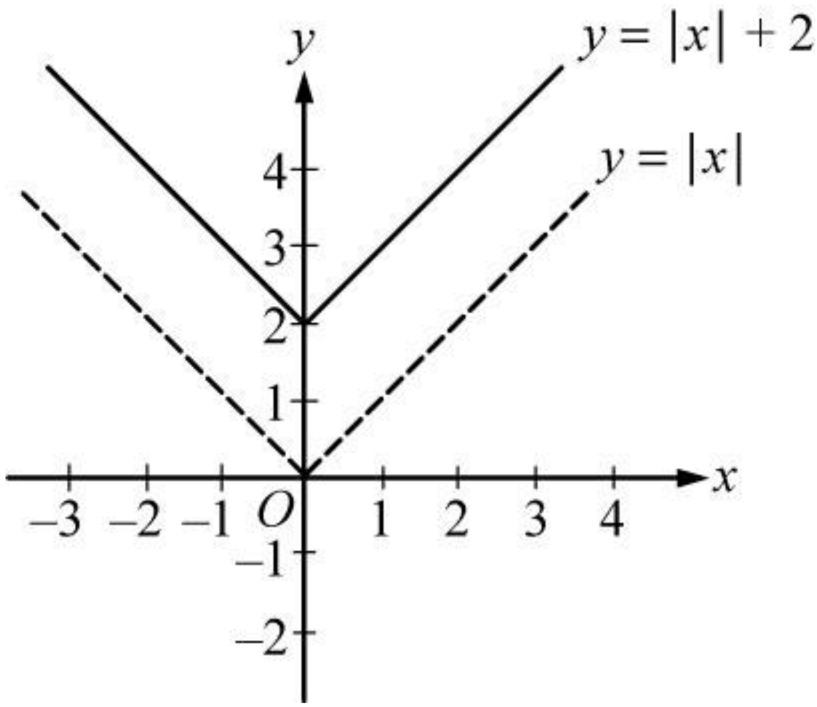
Also note that $y = -\sqrt{x}$ *y* = the negative square root of *x* is the reflection of $y = \sqrt{x}$ *y* = the positive square root of *x* about the *x* axis. In general, for any function *h*, the graph of $y = -h(x)$ *y* = negative *h* of, *x* is the **reflection** of the graph of $y = h(x)$ *y* = *h* of, *x* about the *x* axis.

Example 2.9.5: Consider the functions defined by

$f(x) = |x| + 2$ and $g(x) = (x + 1)^2$. *f* of, *x* = the absolute value of *x*, +, 2 and *g* of, *x* = open parenthesis, *x* + 1, close parenthesis, squared.

These functions are related to the absolute value function $|x|$ absolute value of x and the quadratic function x^2, x squared, respectively, in simple ways.

The graph of $f(x) = |x| + 2$ f of, x = the absolute value of x, +, 2 is the graph of $y = |x|$ y = the absolute value of x shifted upward by 2 units, as shown in Algebra Figure 12 below.



Algebra Figure 12

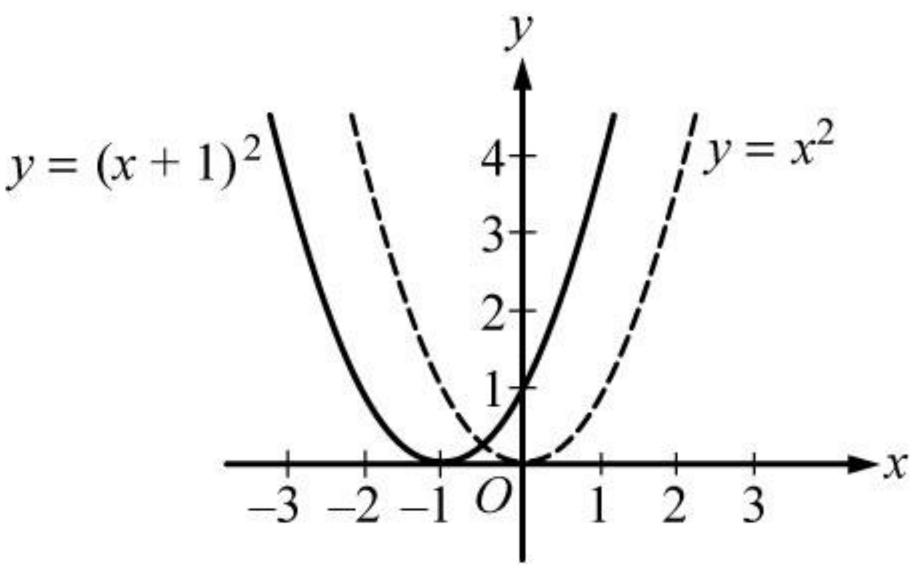
Begin skippable part of description of Algebra Figure 12.

Algebra Figure 12 shows the graph of $y = |x| + 2$ y = the absolute value of x, +, 2 as a solid V shaped curve and the graph of $y = |x|$ y = the absolute value of x as a dashed V shaped curve.

End skippable part of figure description.

Similarly, the graph of the function $k(x) = |x| - 5$ k of, x = the absolute value of x, minus, 2 is the graph of $y = |x|$ y = the absolute value of x shifted downward by 5 units.

The graph of $g(x) = (x+1)^2$ g of, x = open parenthesis, x + 1, close parenthesis, squared is the graph of $y = x^2$ y = x squared shifted to the left by 1 unit, as shown in Algebra Figure 13 below.



Algebra Figure 13

Begin skippable part of description of Algebra Figure 13.

Algebra Figure 13 shows the graph of $g(x) = (x+1)^2$ g of, x = open parenthesis, x + 1, close parenthesis, squared as a solid parabola and the graph of $y = x^2$ y = x squared as a dashed parabola.

End skippable part of figure description.

Similarly, the graph of the function $j(x) = (x-4)^2$ *j* of, *x* = open parenthesis, *x* minus 4, close parenthesis, squared is the graph of $y = x^2$ *y* = *x* squared shifted to the right by 4 units. To double check the direction of the shift, you can plot some corresponding values of the original function and the shifted function.

In general, for any function $h(x)$ *h* of, *x* and any positive number *c*, the following are true.

The graph of $h(x) + c$ *h* of, *x*, +, *c* is the graph of $h(x)$ *h* of *x* **shifted upward** by *c* units.

The graph of $h(x) - c$ *h* of, *x*, minus, *c* is the graph of $h(x)$ *h* of *x* **shifted downward** by *c* units.

The graph of $h(x + c)$ *h* of, the quantity, *x* + *c* is the graph of $h(x)$ *h* of, *x* **shifted to the left** by *c* units.

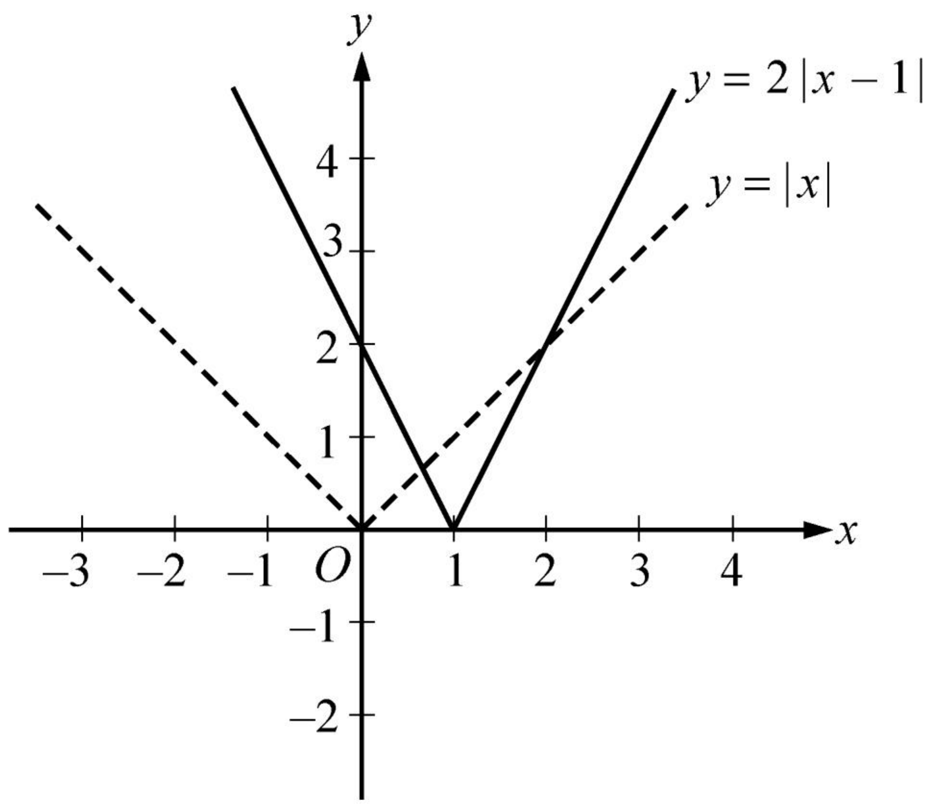
The graph of $h(x - c)$ *h* of, the quantity, *x* minus *c* is the graph of $h(x)$ *h* of, *x* **shifted to the right** by *c* units.

Example 2.9.6: Consider the functions defined by $f(x) = 2|x-1|$ and $g(x) = -\frac{x^2}{4}$.

f of, *x* = 2 times the absolute value of the quantity *x* minus 1, and *g* of, *x* = negative *x* squared, over 4.

These functions are related to the absolute value function $|x|$ absolute value of *x* and the quadratic function x^2, *x* squared, respectively, in more complicated ways than in the preceding example.

The graph of $f(x) = 2|x - 1|$ *f* of, *x* = 2 times the absolute value of the quantity *x* minus 1, is the graph of $y = |x|$ *y* = the absolute value of *x* shifted to the right by 1 unit and then stretched, or dilated, vertically away from the *x* axis by a factor of 2, as shown in Algebra Figure 14 below.



Algebra Figure 14

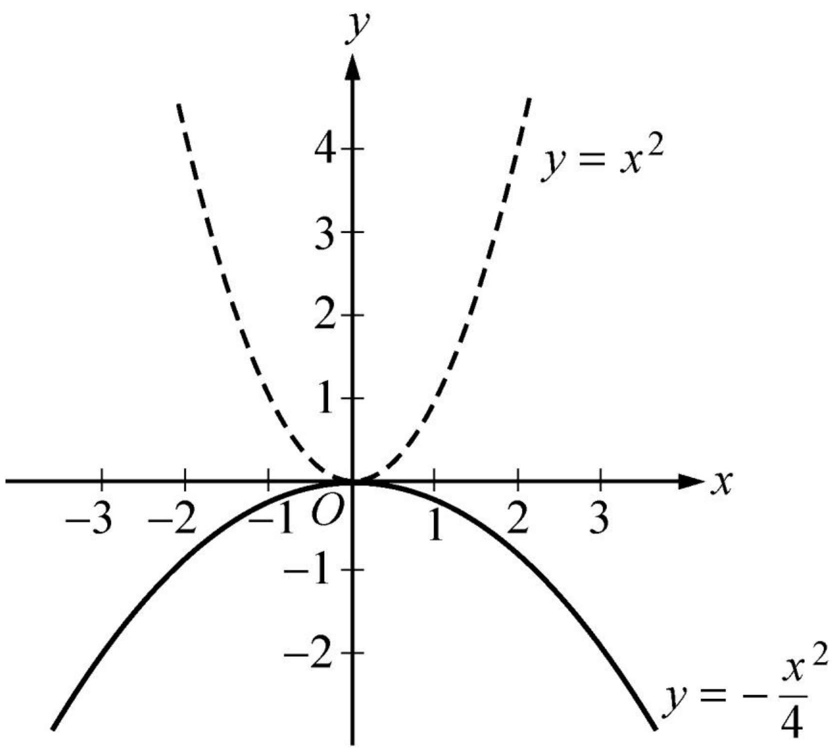
Begin skippable part of description of Algebra Figure 14.

Algebra Figure 14 shows the graph of $f(x) = 2|x - 1|$ *f* of, *x* = 2 times the absolute value of the quantity *x* minus 1 as a solid V shaped curve and the graph of $y = |x|$ *y* = the absolute value of *x* as a dashed V shaped curve. The bottom of the V in the graph of $f(x) = 2|x - 1|$ *f* of, *x* = 2 times the absolute value of the quantity *x* minus 1 is 1 unit to the right of the bottom of the V in the graph of $y = |x|$, y = the absolute value of x, and the V shape of the graph of $f(x) = 2|x-1|$ f of, x = 2 times the absolute value of the quantity x minus 1 is narrower than the V shape of the graph of $y = |x|$. y = the absolute value of x.

End skippable part of figure description.

Similarly, the graph of the function $h(x) = \frac{1}{2}|x-1|$ h of, x = one half times the absolute value of the quantity x minus 1 is the graph of $y = |x|$ y = the absolute value of x shifted to the right by 1 unit and then shrunk, or contracted, vertically toward the x axis by a factor of $\frac{1}{2}$. one half.

The graph of $g(x) = -\frac{x^2}{4}$ g of, x = negative x squared over 4 is the graph of $y = x^2$ y = x squared contracted vertically toward the x axis by a factor of $\frac{1}{4}$ one fourth and then reflected in the x axis, as shown in Algebra Figure 15 below.



Algebra Figure 15

Begin skippable part of description of Algebra Figure 15.

Algebra Figure 15 shows the graph of $g(x) = -\frac{x^2}{4}$ *g* of, *x* = negative *x* squared over 4 as a solid parabola and the graph of $y = x^2$ *y* = *x* squared as a dashed parabola. The graph of $g(x) = -\frac{x^2}{4}$ *g* of, *x* = negative *x* squared over 4 has vertex at the origin, opens downward, and is wider than the parabola $y = x^2$. *y* = *x* squared.

End skippable part of figure description.

In general, for any function $h(x)$ *h* of, *x* and any positive number *c*, the following are true.

The graph of $c\,h(x)$ *c* times *h* of, *x* is the graph of $h(x)$ *h* of, *x* **stretched vertically** by a factor of *c* if $c > 1.$ *c* is greater than 1.

The graph of $c\,h(x)$ *c* times *h* of, *x* is the graph of $h(x)$ *h* of, *x* **shrunk vertically** by a factor of *c* if $0 < c < 1.$ 0 is less than *c*, which is less than 1.

Algebra Exercises

1. Find an algebraic expression to represent each of the following.

a. The square of y is subtracted from 5, and the result is multiplied by 37.

b. Three times x is squared, and the result is divided by 7.

c. The product of $(x+4)$ open parenthesis, *x* + 4, close parenthesis and y is added to 18.

2. Simplify each of the following algebraic expressions.

a. $3x^2 - 6 + x + 11 - x^2 + 5x$ 3, *x* squared, minus 6, +, *x* + 11, minus *x* squared, +, 5*x*

b. $3(5x-1) - x + 4$ 3 times, open parenthesis, 5*x*, minus 1, close parenthesis, minus *x*, +, 4

c. $\frac{x^2 - 16}{x - 4}$, where $x \neq 4$ the expression with numerator *x* squared minus 16 and denominator *x* minus 4, where *x* is not equal to 4

d. $(2x+5)(3x-1)$ open parenthesis, 2*x*, +, 5, close parenthesis, times, open parenthesis, 3*x*, minus 1, close parenthesis

3.

a. What is the value of $f(x) = 3x^2 - 7x + 23$ when $x = -2$? *f* of, *x* = 3, *x* squared, minus 7*x*, +, 23, when *x* is equal to negative 2 ?

b. What is the value of $h(x) = x^3 - 2x^2 + x - 2$ when $x = 2$? *h* of, *x* = *x* cubed, minus 2, *x* squared, +, *x*, minus 2, when *x* = 2 ?

c. What is the value of $k(x) = \frac{5}{3}x - 7$ when $x = 0$? *k* of, *x* = 5 thirds *x*, minus 7, when *x* = 0 ?

4. If the function *g* is defined for all nonzero numbers *y* by $g(y) = \frac{y}{|y|}$, *g* of, *y* = *y* over the absolute value of *y*, find the value of each of the following.

a. $g(2)$ *g* of, 2

b. $g(-2)$ *g* of, negative 2

c. $g(2) - g(-2)$ *g* of, 2, minus, *g* of, negative 2

5. Use the rules of exponents to simplify the following.

a. $(n^5)(n^{-3})$ *n* to the power 5, times, *n* to the power negative 3

b. $(s^7)(t^7)$ *s* to the power 7, times, *t* to the power 7

c. $\frac{r^{12}}{r^4}$ *r* to the power 12 over *r* to the power 4

d. $\left(\frac{2a}{b}\right)^5$ open parenthesis 2*a* over *b*, close parenthesis, to the power 5

e. $\left(w^5\right)^{-3}$ open parenthesis, *w* to the power 5, close parenthesis, to the power negative 3

f. $\left(5^0\right)\left(d^3\right)$ 5 to the power 0, times, *d* to the power 3

g. $\dfrac{\left(x^{10}\right)\left(y^{-1}\right)}{\left(x^{-5}\right)\left(y^5\right)}$

the expression with numerator *x* to the power 10 times *y* to the power negative 1, and denominator *x* to the power negative 5 times *y* to the power 5

h. $\left(\dfrac{3x}{y}\right)^2 \div \left(\dfrac{1}{y}\right)^5$

open parenthesis, 3*x* over *y*, close parenthesis, squared, divided by, open parenthesis, 1 over *y*, close parenthesis, to the power 5

6. Solve each of the following equations for *x*.

a. $5x - 7 = 28$ 5*x*, minus 7 = 28

b. $12 - 5x = x + 30$ 12 minus 5*x* = *x* + 30

c. $5(x + 2) = 1 - 3x$ 5 times, open parenthesis, *x* + 2, close parenthesis, = 1, minus, 3*x*

d. $(x + 6)(2x - 1) = 0$ open parenthesis, *x* + 6, close parenthesis, times, open parenthesis, 2*x*, minus 1, close parenthesis, = 0

e. $x^2 + 5x - 14 = 0$ *x* squared, +, 5*x*, minus 14

f. $x^2 - x - 1 = 0$ *x* squared, minus *x*, minus 1 = 0

7. Solve each of the following systems of equations for x and y.

a. $x + y = 24$
$x - y = 18$

a. $x + y = 24$, and, x minus $y = 18$

b. $3x - y = -5$
$x + 2y = 3$

b. $3x$ minus y = negative 5, and, $x + 2y = 3$

c. $15x - 18 - 2y = -3x + y$
$10x + 7y + 20 = 4x + 2$

c. $15x$ minus 18 minus $2y$ = negative $3x$, +, y, and, $10x$, +, $7y$, +, 20 = $4x$, +, 2

8. Solve each of the following inequalities for x.

a. $-3x > 7 + x$ negative $3x$ is greater than $7 + x$

b. $25x + 16 \geq 10 - x$ $25x + 16$ is greater than or equal to 10 minus x

c. $16 + x > 8x - 12$ $16 + x$ is greater than $8x$, minus 12

9. For a given two digit positive integer, the tens digit is 5 more than the units digit. The sum of the digits is 11. Find the integer.

10. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y ?

11. Kathleen’s weekly salary was increased by 8 percent to $237.60. What was her weekly salary before the increase?

12. A theater sells children’s tickets for half the adult ticket price. If 5 adult tickets and 8 children’s tickets cost a total of $27, what is the cost of an adult ticket?

13. Pat invested a total of $3,000. Part of the money was invested in a money market account that paid 10 percent simple annual interest, and the remainder of the money was invested in a fund that paid 8 percent simple annual interest. If the interest earned at the end of the first year from these investments was $256, how much did Pat invest at 10 percent and how much at 8 percent?

14. Two cars started from the same point and traveled on a straight course in opposite directions for exactly 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed of each car for the 2 hour trip?

15. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, then the cost per person is greater by $12.

a. What is the fixed total cost to charter the aircraft?

b. What is the cost per person if 40 people charter the aircraft?

16. An antiques dealer bought c antique chairs for a total of x dollars. The dealer sold each chair for y dollars.

a. Write an algebraic expression for the profit, P, earned from buying and selling the chairs.

b. Write an algebraic expression for the profit **per chair**.

17. Algebra Figure 16 below shows right triangle *PQR* in the xy plane. Find the following.

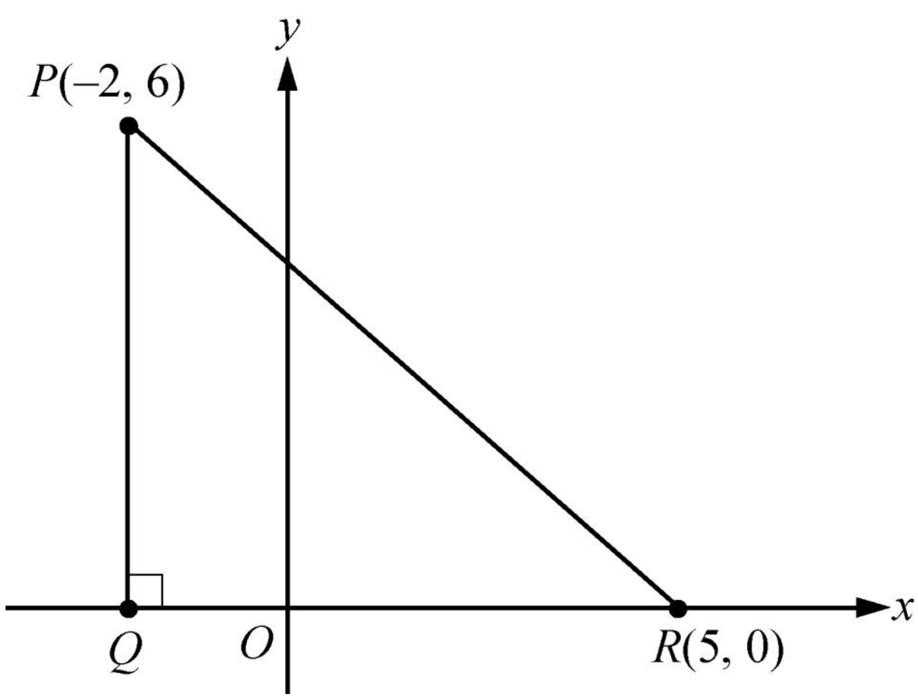
a. Coordinates of point Q

b. Lengths of line segment *PQ*, line segment *QR*, and line segment *PR*

c. Perimeter of triangle *PQR*

d. Area of triangle *PQR*

e. Slope, y intercept, and equation of the line passing through points P and R



Algebra Figure 16

Begin skippable part of description of Algebra Figure 16.

The figure shows right triangle *PQR* in the *x y* plane. Vertex *R* lies on the *x* axis and has coordinates $(5, 0)$. 5 comma 0. Vertex *P* lies in the second quadrant and has coordinates $(-2, 6)$. negative 2 comma 6. Vertex *Q* lies on the *x* axis directly below vertex *P*. Angle *PQR* is a right angle.

End skippable part of figure description.

18. In the *x y* plane, find the following.

a. Slope and *y* intercept of the line with equation $2y + x = 6$

b. Equation of the line passing through the point $(3, 2)$ 3 comma 2 with *y* intercept 1

c. The *y* intercept of a line with slope 3 that passes through the point $(-2, 1)$ negative 2 comma 1

d. The *x* intercepts of the graphs in parts a, b, and c

19. For the parabola $y = x^2 - 4x - 12$ *y* = *x* squared, minus 4*x*, minus 12 in the *x y* plane, find the following.

a. The *x* intercepts

b. The *y* intercept

c. Coordinates of the vertex

20. For the circle $(x-1)^2 + (y+1)^2 = 20$ open parenthesis, *x* minus 1, close parenthesis, squared, +, open parenthesis, *y* + 1, close parenthesis, squared, =, 20 in the *x y* plane, find the following.

a. Coordinates of the center

b. Radius

c. Area

21. For each of the following functions, give the domain and a description of the graph $y = f(x)$ *y* = *f* of, *x* in the *x y* plane, including its shape, and the *x* and *y* intercepts.

a. $f(x) = -4$ *f* of, *x* = negative 4

b. $f(x) = 100 - 900x$ *f* of, *x* = 100 minus 900*x*

c. $f(x) = 5 - (x+20)^2$ *f* of, *x* = 5 minus, open parenthesis, *x* + 20, close parenthesis, squared

d. $f(x) = \sqrt{x+2}$ *f* of, *x* = the positive square root of the quantity *x* + 2

e. $f(x) = x + |x|$ *f* of, *x* = *x* + the absolute value of *x*

Answers to Algebra Exercises

1.

a. $37\left(5-y^2\right)$, or $185-37y^2$

37 times, open parenthesis, 5 minus, *y* squared, close parenthesis, or 185, minus, 37, *y* squared

b. $\frac{(3x)^2}{7}$, or $\frac{9x^2}{7}$

the fraction, open parenthesis, 3*x*, close parenthesis, squared, over 7, or the fraction, 9, *x* squared over 7

c. $18+(x+4)(y)$, or $18+xy+4y$

18 +, open parenthesis, *x* + 4, close parenthesis, times *y*, or, 18 + *x y* + 4*y*

2.

a. $2x^2+6x+5$ 2, *x* squared, + 6*x*, + 5

b. $14x+1$

c. $x+4$

d. $6x^2+13x-5$ 6, *x* squared, + 13*x*, minus 5

3.

a. 49

b. 0

c. -7 negative 7

4.

a. 1

b. -1 negative 1

c. 2

5.

a. n^2 *n* squared

b. $(st)^7$ open parenthesis, *st*, close parenthesis, to the power 7

c. r^8 *r* to the power 8

d. $\frac{32a^5}{b^5}$ the fraction 32, *a* to the power 5, over, *b* to the power 5

e. $\frac{1}{w^{15}}$ 1 over, *w* to the power 15

f. d^3 *d* cubed

g. $\frac{x^{15}}{y^6}$ the fraction *x* to the power 15, over, *y* to the power 6

h. $9x^2y^3$ 9, *x* squared, *y* cubed

6.

a. 7

b. -3 negative 3

c. $-\frac{9}{8}$ negative 9 over 8

d. the two solutions are -6, and $\frac{1}{2}$ negative 6, and one half

e. the two solutions are -7, and 2 negative 7 and 2

f. the two solutions are $\frac{1+\sqrt{5}}{2}$, and $\frac{1-\sqrt{5}}{2}$

the fraction with numerator 1 + the positive square root of 5, and denominator 2, and the fraction with numerator 1 minus the positive square root of 5, and denominator 2

7.

a. $x = 21$; $y = 3$

b. $x = -1$; $y = 2$ *x* = negative 1, *y* = 2

c. $x = \frac{1}{2}$; $y = -3$ *x* = one half; *y* = negative 3

8.

a. $x < -\frac{7}{4}$ *x* is less than negative 7 over 4

b. $x \geq -\frac{3}{13}$ *x* is greater than or equal to negative 3 over 13

c. $x < 4$ *x* is less than 4

9. 83

10. 15 to 8

11. $220

12. $3

13. $800 at 10% and $2,200 at 8%

14. 48 miles per hour and 56 miles per hour

15.

a. $4,320

b. $108

16.

a. $P = cy - x$ *P* = *c y*, minus *x*

b. Profit per chair: $\frac{P}{c} = \frac{cy - x}{c} = y - \frac{x}{c}$

Profit per chair: *P* over *c*, =, the fraction with numerator *c y* minus *x*, and denominator *c*, which is equal to *y* minus the fraction *x* over *c*

17.

a. The coordinates of point *Q* are $(-2, 0)$ negative 2 comma 0

b. The length of *PQ* is 6, the length of *QR* is 7, and the length of *PR* is $\sqrt{85}$. the positive square root of 85.

c. $13 + \sqrt{85}$ 13 + the positive square root of 85

d. 21

e. Slope: $-\frac{6}{7}$; negative 6 over 7; *y* intercept: $\frac{30}{7}$ 30 over 7

equation of line: $y = -\frac{6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

y = negative 6 sevenths *x*, +, 30 over 7, or 7*y* + 6*x* = 30

18.

a. Slope: $-\frac{1}{2}$; negative one half; *y* intercept: 3

b. $y = \frac{x}{3} + 1$ *y* = *x* over 3, +, 1

c. 7

d. $6, -3, \text{ and } -\frac{7}{3}$ 6, negative 3, and negative 7 over 3

19.

a. $x = -2 \text{ and } x = 6$ *x* = negative 2, and *x* = 6

b. $y = -12$ *y* = negative 12

c. $(2, -16)$ coordinates 2 comma negative 16

20.

a. $(1, -1)$ coordinates 1 comma negative 1

b. $\sqrt{20}$ the positive square root of 20

c. 20π 20 pi

21.

a. Domain: the set of all real numbers. The graph is a horizontal line with *y* intercept -4 negative 4 and no *x* intercept.

b. Domain: the set of all real numbers. The graph is a line with slope -900, negative 900, *y* intercept 100, and *x* intercept $\frac{1}{9}$. 1 ninth.

c. Domain: the set of all real numbers. The graph is a parabola opening downward with vertex at $(-20, 5)$, negative 20 comma 5, line of symmetry $x = -20$, *x* = negative 20, *y* intercept -395, negative 395 and *x* intercepts $-20 \pm \sqrt{5}$. negative 20 plus or minus the positive square root of 5.

d. Domain: the set of numbers greater than or equal to -2. negative 2. The graph is half a parabola opening to the right with vertex at $(-2, 0)$, negative 2 comma 0, *x* intercept -2, negative 2, and *y* intercept $\sqrt{2}$. the positive square root of 2.

e. Domain: the set of all real numbers. The graph is two half lines joined at the origin: one half line is the negative *x* axis and the other is a line starting at the origin with slope 2. Every nonpositive number is an *x* intercept, and the *y* intercept is 0. The function is equal to the following piecewise defined function

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

f of, *x* = 2*x*, for *x* greater than or equal to 0, and *f* of, *x* = 0 for *x* less than 0